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# Trade and the Value of Information Under Unawareness\*

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## Abstract

The value of information and the possibility of speculation are examined in an environment with unawareness. Although agents have “correct” prior knowledge about events they are aware of and have a clear understanding of their available actions and payoffs, their unawareness may lead them to commit information processing errors and to behave suboptimally. As a result, more information is not always valuable and agents can speculate with each other. We identify two specific information processing errors that are responsible for both problems. Moreover, we construct a dynamic model where agents announce their posteriors and update their awareness as soon as they hear a counterfactual announcement. We study how awareness is updated and whether agreement about posteriors is reached.

## 1 Introduction

Consider an agent who contemplates investing in the stock market today. His payoff is determined by the prices of shares tomorrow, and his particular buy and sell orders. The agent is aware of all his possible actions (investments) and all the possible prices of the shares.

Suppose now that there are other contingencies, expressed by questions, which indirectly influence the prices of shares tomorrow, and therefore the agent’s payoff. Examples of such questions are whether there will be a merger, what are the characteristics of a new CEO or whether an innovation will be announced. The agent may be aware of some of these questions, and unaware of others. Being unaware of a question means that he does not know its answer and he does not know that he does not know. In other words, he misses some information and at the same time fails to recognize it. The agent receives some information and some awareness before he chooses his action.

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Although the agent has an *incomplete* understanding of the world and there are questions that he has never thought about, he nevertheless has correct prior knowledge about events he *is aware of* and, within the bounds of his reasoning, he is perfectly rational. He is aware of all of his possible actions and he does not err when contemplating their deterministic payoff.

In such an environment is information valuable? That is, will the more informed agent be better off *ex ante* when compared with a less informed but otherwise identical agent? In an interactive setting, will agents who share a common prior speculate against each other? Finally, do agents update their awareness and how?

As subsequent examples show, the value of information can be negative and agents can speculate. Since unawareness is a mistake of reasoning, this is not surprising. Even within the context of the standard model of knowledge both of these phenomena can occur when agents have a non partitional information structure. However, modelling unawareness explicitly allows us to analyze the nature of these mistakes and distinguish between those that agents are most likely to commit systematically (because of being unaware), from those that are random or of no economic interest (like mistakes in computing one's payoff).

We identify two mistakes in reasoning. In order to understand the first, we need to interpret awareness as a signal. Depending on which state occurs, the agent receives some information and awareness signals. Before choosing his action, he excludes all states which specified different information and/or lower awareness. However, he cannot exclude states describing higher awareness, since he cannot reason beyond his current level. In other words, he uses his "awareness" signal in an asymmetric way and this may lead to suboptimal choices.

The second mistake arises when the level of one's awareness is too low. A low level of awareness means that even though the agent is more informed about events he is aware of, there are many events that he is unaware of. As he cannot condition his actions on events he is unaware of, he is more inflexible and this comes at a cost. In an interactive setting, a low level of awareness implies that the agent may make wrong inferences about the information and actions of others. This is because the agent cannot reason beyond his awareness and as a result he may miss some connections (theorems) between events he *is aware of* and events he *unaware of*. Unavoidably, he will also be unaware that others actually know these theorems and therefore underestimate their knowledge.

This mistake in reasoning can lead to nasty surprises. Suppose that agent 1 is certain that agent 2 considers a high and a low price equally likely and therefore he should not offer to buy or sell. How he should react if agent 2 offers to buy? Since the only source of wrong reasoning about others is his low awareness, he may attempt to increase or update his own. By constructing a dynamic model where agents announce their posteriors about an event, we study how awareness is updated and whether agreement about posteriors (and actions) is reached.

## 1.1 Related Literature

The value of information has been studied in a variety of settings. [Blackwell \(1951\)](#) shows that an experiment is more informative if and only if it is more valuable to the decision maker. Blackwell’s theorem fails when the agent is not an expected utility maximizer, as shown by [Safra and Sulganik \(1995\)](#), [Schlee \(1991\)](#) and [Wakker \(1988\)](#).

The setting that is closest to that of the present paper represents information as a partition of the state space. If the agent’s prior is correct, he updates using Bayes’ rule and chooses an action which maximizes his expected utility, then more information, measured by a finer partition, makes him better off ex ante. However, a partition is consistent only with an unboundedly rational agent who makes no information processing errors.

[Sher \(2005\)](#) shows that if the agent has wrong priors and updates nonmonotonically, then a little more information can be bad, but a lot of information is always good. [Geanakoplos \(1989\)](#) shows that when the agent has correct priors but a non-partitional information structure, then it is not always good to know more, unless this structure satisfies non-delusion, positive introspection, and nested.

Geanakoplos provides several reasons why agents in his model make information processing errors. For example, agents may forget, ignore unpleasant information, take no notice when something does not occur or be unaware. However, as demonstrated by [Dekel et al. \(1998\)](#), his setting has some limitations if the intention is to model agents who make information processing errors because of their unawareness.<sup>1</sup> Another limitation is that Geanakoplos only compares an agent who makes some information processing errors with another agent who is less informed but makes no errors - so his information is represented by a partition. In this paper we take a more general approach and allow for any agent, not just the more informed, to be unaware and to make information processing errors.

By applying the unawareness model provided by [Galani \(2007\)](#), we are able to specify what are the mistakes in reasoning that can result in speculative trade and in negative value of information and address the criticism of [Dekel et al. \(1998\)](#).

The literature on no trade theorems stems from [Aumann \(1976\)](#). Agents trade either because they have different priors or because they make information processing errors. In the context of the standard model where agents make no mistakes, [Morris \(1994\)](#), [Feinberg \(1995, 1996\)](#), [Samet \(1998\)](#), [Bonanno and Nehring \(1996\)](#) and [Ng \(2003\)](#) show that “a necessary and sufficient condition for the existence of a common prior is that there is no bet for which it is always common knowledge that all agents expect a positive gain”. Moreover, [Milgrom and Stokey \(1982\)](#) and [Sebenius and Geanakoplos \(1983\)](#) show that common priors imply that there cannot be common knowledge of speculation. [Geanakoplos and Polemarchakis \(1982\)](#) show that if agents take turns in announcing their posteriors about an event, they will eventually agree. [Geanakoplos \(1989\)](#) shows that if we allow for mistakes, agents can speculate, even with common priors. However, speculation in equilibrium cannot occur, as long as non-delusion, nested and positive introspection are satisfied.

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<sup>1</sup>[Dekel et al. \(1998\)](#) show that if unawareness satisfies three properties they propose, then the standard state space model can only accommodate trivial unawareness.

Models of unawareness are either syntactic or semantic. Beginning with [Fagin and Halpern \(1988\)](#), syntactic models have been provided by [Halpern \(2001\)](#), [Modica and Rustichini \(1994, 1999\)](#), [Halpern and Rêgo \(2005\)](#), [Heifetz et al. \(2008a\)](#), [Board and Chung \(2006\)](#) and [Galanis \(forthcoming\)](#). [Geanakoplos \(1989\)](#) provides one of the first set-theoretic models of boundedly rational agents, using the standard framework. [Dekel et al. \(1998\)](#) argues against the use of a standard state space by proposing three properties for an unawareness operator. Two approaches overcome this impossibility result. First, by arguing against one of the properties ([Ely \(1998\)](#)), or by relaxing them ([Xiong \(2007\)](#)). Second, by introducing multiple state spaces, as in [Li \(2009\)](#), [Heifetz et al. \(2006\)](#) and [Galanis \(2007\)](#).

The main difference between the model of [Galanis \(2007\)](#) and those of [Heifetz et al. \(2006\)](#) is that the former allows for knowledge and posteriors to vary across states that differ only with respect to how expressive they are in terms of awareness.<sup>2</sup> This difference is crucial in the applications we consider here because we make the assumption that agents are aware of and make no mistakes about their actions and payoffs. Moreover, we concentrate on how agents reason about events that everyone is aware of. For example, we only study betting about events that everyone is aware of. If being less aware did not allow for incorrect reasoning about knowledge and posteriors, then all the results from the standard model would be true here as well.

Applications in the context of games with unawareness have been provided by [Feinberg \(2004, 2005\)](#), [Čopić and Galeotti \(2007\)](#), [Li \(2006b\)](#), [Heifetz et al. \(2008b\)](#), [Heifetz et al. \(2009\)](#), [Filiz-Ozbay \(2008\)](#), [Ozbay \(2008\)](#), [von Thadden and Zhao \(2008\)](#) and [Halpern and Rêgo \(2006\)](#).

## 1.2 Overview of the results

The value of information is analyzed by comparing a more informed agent 2 with a less informed agent 1. Both have the same preferences, payoffs, prior, and a correct understanding of their payoffs and actions, but may differ in their awareness and information. In the context of the standard model, agent 2 is more informed than agent 1 if whenever 1 knows an event, 2 knows it as well.

Information is valuable if both mistakes in reasoning are addressed. The low level of awareness is addressed by requiring that whenever agent 1 knows an event  $E$  that 2 is unaware of, agent 2 knows another event that logically implies  $E$ . Whenever this happens, we say that 2 is strongly more informed.

For the second mistake we need two properties. The first, nested awareness, requires that there exists an ordering of all questions and each agent is aware of a question only if he is also aware of all the other questions that precede it, according to this order.

The second property, conditional independence, requires that awareness, as a signal, has no informational value to the decision maker. Conditional independence can be phrased in terms of an example. Suppose that an agent who considers investing in a firm were to acquire some private information, that enabled him to consider only two mutually exclusive scenarios as possible. Either there will be a lawsuit against the firm, or the firm will announce a technological breakthrough. Conditional independence

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<sup>2</sup>For details on this difference see [Galanis \(2007\)](#).

requires that the likelihood that his awareness will increase is not influenced by whether the breakthrough or the lawsuit occur. In other words, there is nothing intrinsic in one of the two events that can change the way his awareness varies. For example, the property would fail if an innovation meant that the agent most likely becomes aware of a new dimension, whereas this does not happen with a lawsuit. Hence, we would expect conditional independence to hold when it is “business as usual” for investors, and to fail otherwise.

Information is valuable if one of the following is true. First, the strongly more informed agent 2 satisfies conditional independence, so that he does not misuse his awareness signal. Second, agent 1 satisfies conditional independence and the strongly more informed agent 2’s awareness is nested and more informative.

Trade is analyzed in three different settings. Suppose that all agents share a common prior. Then, there cannot be common knowledge trade. Hence, although agents may be boundedly rational due to their unawareness, common knowledge of trade is sufficiently strong to rule out any such possibility, just like in the standard model with partitional structures. This result is in contrast with models of bounded reasoning (Geanakoplos (1989)) and unawareness (Xiong (2007)) that use the standard framework but non-partitional structures, as common knowledge trade is feasible there.

This implies that assigning zero probability to an event is behaviorally distinct from being unaware of it. Agents who assign probability zero to some events would engage in common knowledge knowledge trade, while unaware agents would not. Interestingly, we also show that common knowledge of *no trade* does not imply that there is no trade, the reason being that agents may make wrong inferences about others, due to their low level of awareness. Hence, although agents cannot agree that there are unexploited trading opportunities, such opportunities may exist, as long as they are beyond the awareness of some.

Second, conditional independence implies that there does not exist a trade that provides positive expected gains for everyone, always. That is, as long as agents do not overestimate or underestimate events due to their varying awareness, a mutually beneficial trade does not exist. Moreover, the reverse does not hold, so that even with conditional independence, different priors do not imply the existence of a mutually beneficial trade. The problem of low awareness is not relevant in this setting because agents do not need to reason about the actions of others.

Trade in equilibrium cannot occur if each agent satisfies either conditional independence or nested awareness, and his payoff is not influenced by the level of his awareness. The second condition addresses the issue that, in equilibrium, agents have to reason about the actions of others. Low level of awareness means that they may reason incorrectly, so their actual payoffs may be different from their perceived ones, if this condition is not satisfied.

In the last part of the paper we study how agents update their awareness, by constructing a dynamic model where two agents take turns in announcing their posteriors. We show that agents eventually agree on their posteriors, just like in the standard model. Finally, we place lower and upper bounds on how much awareness is updated. First, the agent updates at all if and only if he becomes aware of something that the other is aware of. Second, (under a mild condition) the agent can at most become

aware of everything that the other agent is already aware of. Hence, announcements transmit awareness between agents.

The paper proceeds as follows. Section 2 provides an example showing that information is not always valuable in the presence of unawareness. Section 3 presents an overview of the model of unawareness of Galanis (2007) and formalizes the conditions mentioned above. The value of information problem is analyzed in section 4, whereas the no trade theorems are presented in section 5. In section 6 a dynamic version of the model is constructed in order to analyze how agents update their awareness. Proofs are contained in the appendix.

## 2 Knowing less can be better

In the following example we show that the value of information can be negative in an environment with unawareness. In particular, we compare two agents who share the same awareness, payoffs, action set and prior, but one has more information and he is strictly worse off.

Everything that is relevant about the world can be described by giving an answer to the following three questions. Question  $p$ , “What is the price of the share?”, has three possible answers: low, medium and high. Question  $q$ , “Is there going to be an acquisition?”, and question  $r$ , “Is there an innovation going to be adopted?”, both have two possible answers, “yes” and “no”.

There are two available actions, buy (B) and not buy (NB). Payoffs depend only on the price. More specifically, the payoff (in utils) is zero if the action is NB, irrespective of the price. If the action is B then the payoff is 1 if the price is high and  $-1$  otherwise. Therefore, in terms of payoffs there is no difference between a low and a medium price. One can think of a low and a medium price as two distinct bad scenarios.

A *full state*  $\omega^*$  specifies an answer to all three questions and thus provides a complete description of the world. Let  $\Omega^*$  denote the *full state space*, the collection of full states, and let  $\pi$  be a prior on  $\Omega^*$ . The full state space contains the following three full states:

$$\begin{aligned}\omega_1^* &= (p_l, q_y, r_n), & \pi(\omega_1^*) &= 0.3, \\ \omega_2^* &= (p_m, q_n, r_y), & \pi(\omega_2^*) &= 0.3, \\ \omega_3^* &= (p_h, q_y, r_y), & \pi(\omega_3^*) &= 0.4.\end{aligned}$$

Note that although payoffs depend only on prices, information about the other two questions can help an agent make a better decision. For example, if an agent knew that there is no acquisition ( $q_n$ ), he would understand that the price is medium ( $p_m$ ) and therefore he would choose NB.

Suppose now that both agents are always aware of questions  $p$  and  $q$ , while they are aware of question  $r$  if and only if state  $\omega_3^*$  occurs. Being unaware of question  $r$  means that you do not know whether there is an innovation, and you do not know that you do not know. In other words, the agent completely misses the  $r$  dimension and his view of the world is represented by the following state space  $\Omega$ :

$$\omega_1 = (p_l, q_y), \quad \omega_2 = (p_m, q_n), \quad \omega_3 = (p_h, q_y).$$

However, an unaware agent misses much more than just information about dimension  $r$ . The full state space  $\Omega^*$  specifies that the agent is aware of  $r$  if and only if the price is high. This “fact” is lost in state space  $\Omega$ , because dimension  $r$  is absent. In other words, at  $\omega_3^*$ , a fully aware agent can reason as follows: “I am aware of all three questions. Hence, the price is high”. But at  $\omega_1^*$ , an agent who is unaware of  $r$  *cannot* reason as follows: “I am unaware of  $r$ . Hence, the price cannot be high”. It turns out that this asymmetry drives the result that information may not be valuable.

Effectively, when the agent’s awareness varies across states, it creates a signal that the agent can only understand partially. At  $\omega_3^*$ , the awareness signal partitions the payoff relevant state space as follows:  $(p_l, p_m), (p_h)$ . But at  $\omega_1^*, \omega_2^*$ , the awareness signal provides the trivial partition  $(p_l, p_m, p_h)$ .

Although the two agents are identical in terms of their awareness, payoffs, priors and actions, they have different information. Suppose that agent 2 has a signal that always provides an answer to question  $q$ . Hence, he is more informed than agent 1. The following table summarizes the information of each agent about prices at each full state.

Full state	Agent 1	Agent 2
$\omega_1^*$	$\{p_l, p_m, p_h\}$	$\{p_l, p_h\}$
$\omega_2^*$	$\{p_l, p_m, p_h\}$	$\{p_m\}$
$\omega_3^*$	$\{p_h\}$	$\{p_h\}$

Note that at  $\omega_3^*$  both agents use their awareness signal and deduce that the price is high. At  $\omega_1^*$ , the less informed agent cannot exclude any states, whereas agent 2 can use his information signal and exclude that price is medium, since he knows that there is an acquisition.

Let  $\pi$  be a prior on the full state space  $\Omega^*$ , such that  $\pi(\omega_1^*) = \pi(\omega_2^*) = 0.3$  and  $\pi(\omega_3^*) = 0.4$ . We assume that whenever the agent has a lower dimensional subjective state space, his prior is the marginal of  $\pi$  on that state space.<sup>3</sup>

At full state  $\omega^*$  each agent receives his awareness, he constructs his subjective state space and receives his information. Having a prior on his subjective state space he updates using Bayes’ rule and chooses an action that maximizes his expected utility, given his information. The following table summarizes each agent’s posteriors about the payoff relevant state space  $(p_l, p_m, p_h)$  and best action at each full state. For instance, at  $\omega_1^*$  agent 2 assigns posterior knowledge of 3/7 to the price being low, 4/7 to the price being high and his best action is to buy.

Full state	Agent 1	Action	Agent 2	Action
$\omega_1^*$	(0.3, 0.3, 0.4)	NB	(3/7, 0, 4/7)	B
$\omega_2^*$	(0.3, 0.3, 0.4)	NB	(0, 1, 0)	NB
$\omega_3^*$	(0, 0, 1)	B	(0, 0, 1)	B

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<sup>3</sup>Every subjective state  $\omega$  gives an answer to *some* questions. We assume that the probability assigned to  $\omega$  is the probability that  $\pi$  assigns to the set of full states which give the same answers as  $\omega$ . For example, the probability assigned to  $\omega_1 = (p_l, q_y)$  is  $\pi(\omega_1) = 0.3$ , because  $\omega_1^* = (p_l, q_y, r_n)$  is the only full state which gives the same answers as  $\omega_1$  and  $\pi(\omega_1^*) = 0.3$ . Generally, many full states project to each state  $\omega$ .

The less informed agent 1 always chooses the correct action, since he buys only when the price is high. On the contrary, the more informed agent 2 makes the wrong choice of buying when the price is low. The reason is the asymmetry of the awareness signal: agent 2 is able to use it at  $\omega_3^*$  but not at  $\omega_1^*$ . If the agent was never allowed to use the signal or he was able to use it always, then more information would always be beneficial.

## 3 The Model

### 3.1 Preliminaries

This section presents a reduced version of the model developed in Galanis (2007). In order to examine the value of information and the updating of awareness we only need the possibility correspondences  $P^i$  and therefore we do not define the knowledge and awareness operators. Hence, the only difference with the model of Heifetz et al. (2006) is that their property Projections Preserve Knowledge is not assumed here.<sup>4</sup>

Consider a complete lattice of disjoint state spaces  $\mathcal{S} = \{S_a\}_{a \in A}$  and denote by  $\Sigma = \cup_{a \in A} S_a$  the union of these state spaces. A state  $\omega$  is an element of some state space  $S$ . Let  $\preceq$  be a partial order on  $\mathcal{S}$ .  $(\mathcal{S}, \preceq)$  is well-founded, so that any non-empty subset  $X$  of  $\mathcal{S}$  contains a  $\preceq$ -minimal element.<sup>5</sup> For any  $S, S' \in \mathcal{S}$ ,  $S \preceq S'$  means that  $S'$  is more expressive than  $S$ . Moreover, there is a surjective projection  $r_S^{S'} : S' \rightarrow S$ . Projections are required to commute. If  $S \preceq S' \preceq S''$  then  $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$ . If  $\omega \in S'$ , denote  $\omega_S = r_S^{S'}(\omega)$  and  $\omega^{S''} = \{\omega' \in S'' : r_{S'}^{S''}(\omega') = \omega\}$ . If  $E \subseteq S'$ , denote by  $E_S = \{\omega_S : \omega \in E\}$  the restriction of  $E$  on  $S$  and by  $E^{S''} = \cup\{\omega^{S''} : \omega \in E\}$  the enlargement of  $E$  on  $S''$ . Let  $g(S) = \{S' : S \preceq S'\}$  be the collection of state spaces that are at least as expressive as  $S$ . For a set  $E \subseteq S$ , denote by  $E^\dagger = \cup_{S' \in g(S)} E^{S'}$  the enlargements of  $E$  to all state spaces which are at least as expressive as  $S$ . Let  $r_S^S$  be the identity for any  $S \in \mathcal{S}$ .

Consider a possibility correspondence  $P^i : \Sigma \rightarrow 2^\Sigma \setminus \emptyset$  with the following properties:

- (0) Confinedness: If  $\omega \in S$  then  $P^i(\omega) \subseteq S'$  for some  $S' \preceq S$ .
- (1) Generalized Reflexivity:  $\omega \in (P^i(\omega))^\dagger$  for every  $\omega \in \Sigma$ .
- (2) Stationarity:  $\omega' \in P^i(\omega)$  implies  $P^i(\omega') = P^i(\omega)$ .
- (3) Projections Preserve Ignorance: If  $\omega \in S'$  and  $S \preceq S'$  then  $(P^i(\omega))^\dagger \subseteq (P^i(\omega_S))^\dagger$ .
- (4) Projections Preserve Awareness: If  $\omega \in S'$ ,  $\omega \in P^i(\omega)$  and  $S \preceq S'$  then  $\omega_S \in P^i(\omega_S)$ .

### 3.2 Events and common knowledge

Formally, an *event* is a pair  $(E, S)$ , where  $E \subseteq S$  and  $S \in \mathcal{S}$ . The negation of  $(E, S)$ , defined by  $\neg(E, S) = (S \setminus E, S)$ , is the complement of  $E$  with respect to  $S$ . Let

<sup>4</sup>For a comparison of the present model and that of Heifetz et al. (2006), see Galanis (2007).

<sup>5</sup>This means that there is a  $S \in X$  such that, for all  $S' \in X$ , if  $S' \preceq S$  then  $S' = S$ .

$\mathcal{E} = \{(E, S) : E \subseteq S, S \in \mathcal{S}\}$  be the set of all events. We write  $E$  as a shorthand for  $(E, S)$  and  $\emptyset_S$  as a shorthand for  $(\emptyset, S)$ . For each event  $E$ , let  $S(E)$  be the state space of which it is a subset. An event  $E$  “inherits” the expressiveness of the state space of which it is a subset. Hence, we can extend  $\preceq$  to a partial order  $\preceq_0$  on  $\mathcal{E}$  in the following way:  $E \preceq_0 E'$  if and only if  $S(E) \preceq S(E')$ . Abusing notation, we write  $\preceq$  instead of  $\preceq_0$ .

Let  $\Omega^i : \Sigma \rightarrow \mathcal{S}$  be such that for any  $\omega \in \Sigma$ ,  $\Omega^i(\omega) = S$  if and only if  $P^i(\omega) \subseteq S$ .  $\Omega^i(\omega)$  denotes the most expressive universal event that the agent is aware of at  $\omega$ . We can therefore interpret  $\Omega^i(\omega)$  as agent  $i$ 's state space at  $\omega$ . If  $\Omega^i(\omega) \succ \Omega^j(\omega)$  then we say that  $i$  is more aware than  $j$  at  $\omega$ .

In order to define common knowledge let  $E'$  be a set of states and let  $P^i(E') = \bigcup_{\omega' \in E'} P^i(\omega')$  be the set of states that  $i$  considers possible if the truth lies in  $E'$ .

Suppose  $E$  is an event and  $E'$  is a set of states (not necessarily an event). Slightly abusing notation, write  $E' \succeq E$  if for every state  $\omega \in E'$ ,  $\{\omega\} \succeq E$ . Moreover, if  $E' \succeq S$ , write  $E'_S = \{\omega_S : \omega \in E'\}$  for the set of states that project to  $S$ .<sup>6</sup>

**Definition 1.** *Event  $E \preceq S$  is common knowledge among agents  $i = 1, \dots, I$  at  $\omega \in S$  if and only if for any  $n \in \mathbb{N}$  and any sequence of agents  $i_1, \dots, i_n$ ,  $P^{i_n} \dots P^{i_1}(\omega) \succeq E$  and  $(P^{i_n} \dots P^{i_1}(\omega))_{S(E)} \subseteq E$ .*

In words,  $E$  is common knowledge if for all sequences of agents, in all the states that  $i_1$  considers possible that  $i_2$  considers possible that  $\dots i_{n-1}$  considers possible,  $i_n$  is aware of  $E$  and all states that he considers possible, when projected to state space  $S(E)$ , are contained in  $E$ . This is a generalization of a definition of common knowledge in the standard model: Event  $E$  is common knowledge at  $\omega$  if for every sequence of agents,  $P^{i_n} \dots P^{i_1}(\omega) \subseteq E$ .<sup>7</sup>

### 3.3 Information

In the standard model, more information is represented by a finer partition. Formally, agent 2 is more informed than agent 1 if  $P^2(\omega) \subseteq P^1(\omega)$  for all  $\omega \in \Omega$ . This is equivalent to requiring that whenever agent 1 knows an event, agent 2 knows it as well, so that  $K^1(E) \subseteq K^2(E)$  for all events  $E \subseteq \Omega$ , where  $K$  is the standard knowledge operator.

In an environment with unawareness agents can have different information *and* different awareness. In order to disentangle information from awareness we will consider two properties. The first compares the agents' partitions in the “highest” state space that both are aware of.

**Definition 2.**  *$P^2$  is weakly more informed than  $P^1$  if  $P^2(\omega^*)_{S_{\omega^*}} \subseteq P^1(\omega^*)_{S_{\omega^*}}$  for all  $\omega^* \in S^*$ , where  $S_{\omega^*} = \Omega^1(\omega^*) \wedge \Omega^2(\omega^*)$ .*

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<sup>6</sup>Note that  $P^i(E') = \bigcup_{\omega \in E'} P^i(\omega)$  is not necessarily an event.

<sup>7</sup>For details, see [Geanakoplos \(1992\)](#). For a more detailed exposition of knowledge and common knowledge in an environment with unawareness, see [Galanis \(2007\)](#).

Recall that  $P^2(\omega^*)_{S_{\omega^*}}$  is the projection of 2's information at  $\omega^*$  to state space  $S_{\omega^*}$ , which is the meet (greatest lower bound) of the agents' state spaces at  $\omega^*$ . Hence, this property compares the agents' knowledge only about events that both are aware of.

The second property compares the agents' information in the full (most complete) state space  $S^*$ .

**Definition 3.**  $P^2$  is strongly more informed than  $P^1$  if  $P^2(\omega^*)^{S^*} \subseteq P^1(\omega^*)^{S^*}$  for all  $\omega^* \in S^*$ .

Recall that  $P^2(\omega^*)^{S^*}$  is the enlargement (opposite of projection) of 2's information at  $\omega^*$  to the full state space. It is straightforward to show that strongly more informed implies weakly more informed.

### 3.4 Awareness

Here we group three properties about the agent's awareness. The first requires that awareness is ordered. Formally, an agent's awareness at a full state is either more or less expressive than his awareness at any other full state.

**Definition 4. [Nested awareness]** Awareness for  $P$  is nested if for all  $\omega^*, \omega_1^* \in S^*$ , either  $\Omega(\omega^*) \preceq \Omega(\omega_1^*)$  or  $\Omega(\omega_1^*) \preceq \Omega(\omega^*)$ .

This is equivalent to having a partial order on the collection of all state spaces, so that the agent is aware of a state space  $S$  only if he is also aware of all states spaces  $S'$  that precede it, according to this order. Nested awareness is closely related to the property nested, discussed in [Geanakoplos \(1989\)](#).

For the next two properties we need the following definition. Let

$$\mathcal{E}(\omega^*) = \{\omega_1^* \in S^* : \Omega(\omega^*) = \Omega(\omega_1^*)\}$$

be the event of the full state space describing that the agent has the same awareness as  $\omega^*$  describes. Function  $\mathcal{E}$  partitions the full state space  $S^*$  and provides a signal that the agent can only partially comprehend. If one's awareness varies a lot across states, then his signal is more informative (the partition on  $\Omega^*$  generated by  $\mathcal{E}$  is finer). However, this does not mean that he can “use” this information properly since he only understands the signal partially, as he is not aware of the full state space. This notion is formalized by the following definition.

**Definition 5.** Awareness for  $P^1$  is more informative than for  $P^2$  if for all  $\omega^* \in S^*$ ,  $\mathcal{E}^1(\omega^*) \subseteq \mathcal{E}^2(\omega^*)$ .

The following property requires that conditional on receiving one's private information,  $P(\omega^*)$ , the event describing the agent's awareness,  $\mathcal{E}(\omega^*)$ , is independent of any other event he is aware of.

**Definition 6. [Conditional independence]**  $(P, \pi)$  satisfies conditional independence if for any  $\omega^* \in S^*$  with  $\pi(\omega^*) > 0$ , for any  $E \subseteq \Omega(\omega^*)$ ,

$$\pi(E^{S^*} | P(\omega^*)^{S^*}) = \pi(E^{S^*} | \mathcal{E}(\omega^*) \cap P(\omega^*)^{S^*}).^8$$

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<sup>8</sup>The set  $E^{S^*}$  contains all full states that project to  $E$ , while  $P(\omega^*)^{S^*}$  contains all full states that project to  $P(\omega^*)$ . Hence,  $E^{S^*}$  is the event “ $E$  occurs”, while  $P(\omega^*)^{S^*}$  is the event “ $P(\omega^*)$  occurs”. The equality is equivalent to  $\pi(\mathcal{E}(\omega^*) \cap E^{S^*} | P(\omega^*)^{S^*}) = \pi(\mathcal{E}(\omega^*) | P(\omega^*)^{S^*}) \pi(E^{S^*} | P(\omega^*)^{S^*})$ .

Conditional independence specifies that awareness, as a signal, has no informational value. In other words, knowledge conditional on the information signal  $P$  are identical to knowledge conditional on  $P$  and the awareness signal  $\mathcal{E}$ . Conditional independence implies that an unaware agent cannot misuse his awareness signal, because it provides no information at all. Note that nested awareness and conditional independence use the full state space  $S^*$ . Hence, the agent cannot, in principle, check whether he satisfies these conditions. Moreover, if the agent's awareness is constant across states, then both conditional independence and nested awareness are satisfied.

## 4 The value of information

Let  $C$  be a set of possible actions and define  $u : C \times \Sigma \rightarrow \mathbb{R}$  to be the agent's utility function. When the agent is aware of a state  $\omega$ , his perception of his payoff at  $\omega$  if he chooses action  $c$  is  $u(c, \omega)$ . But  $\omega$  may be a coarse description of the world. We will assume that the agent does not err when contemplating the consequences of his actions. That is, he is always aware of the payoff relevant state space. Let  $S_0 = \bigwedge \mathcal{S}$  be the meet of all the agent's subjective states.

**Assumption 1.** For all  $\omega \in \Sigma$ ,  $u(c, \omega_{S_0}) = u(c, \omega)$  for all  $c \in C$ .

The assumption states that in terms of payoffs, only  $S_0$  matters. Hence, if two states  $\omega, \omega'$  project to the same state  $\omega'' \in S_0$ , (i.e.  $\omega_{S_0} = \omega'_{S_0} = \omega''$ ) then they assign the same payoff for every action. Since the agent is always aware of  $S_0$ , he has a correct understanding of his payoffs for each action. This assumption is consistent with the story, outlined in the example, of an agent who invests in the stock market and has a clear understanding of the payoffs of his buy and sell orders, as they only depend on prices. Although there are many factors that could influence his payoffs (a merger, an innovation, a lawsuit), this can only happen through prices. This assumption is similar to that made in [Morris \(1994\)](#), where signals are not payoff relevant.

A decision problem is a tuple  $(C, \Sigma, P, u, \pi)$  where  $C$  is the action set,  $\Sigma$  is the union of all state spaces,  $P$  is the agent's possibility correspondence,  $u$  is his utility function and  $\pi$  is a prior on  $S^*$ .

Suppose that the agent is partially aware and his subjective state space is  $\Omega$ . We will assume that although the agent is unaware of  $S^*$  and of  $\pi$ , he nevertheless has correct knowledge about events that he is aware of. If he had "wrong" knowledge, then more information could harm him even in the absence of unawareness. Hence, we require that his prior on  $\Omega$  is the marginal of  $\pi$  on  $\Omega$ , denoted by  $\pi|_{\Omega}$ . For every  $\omega \in \Omega$ ,  $\pi|_{\Omega}(\omega) = \sum_{\omega^* \in \omega^{S^*}} \pi(\omega^*)$ , where  $\omega^{S^*}$  is the set of full states that project to  $\omega$ .

A decision function  $f$  maps each full state  $\omega^* \in S^*$  to a specific action  $c \in C$ . The agent chooses his best action by maximizing his expected value, given his information.

**Definition 7.** A decision function  $f : S^* \rightarrow C$  is optimal for the decision problem  $(C, \Sigma, P, u, \pi)$  if and only if

1. For all  $\omega^*, \omega_1^* \in S^*$ ,  $P(\omega^*) = P(\omega_1^*) \implies f(\omega^*) = f(\omega_1^*)$ .

2. For all  $\omega^* \in S^*$  and  $c \in C$ ,

$$\sum_{\omega \in P(\omega^*)} u(f(\omega^*), \omega) \pi|_{\Omega(\omega^*)}(\omega) \geq \sum_{\omega \in P(\omega^*)} u(c, \omega) \pi|_{\Omega(\omega^*)}(\omega).$$

This is a straight generalization of the definition used in [Geanakoplos \(1989\)](#). The first condition states that if two full states describe the same awareness and knowledge, then the agent chooses the same action. The second condition says that once the agent receives his awareness and his information, he updates his knowledge using Bayes' law and chooses the action that maximizes his expected utility given his information. A decision problem is more valuable if each decision function guarantees a higher ex ante expected utility.

**Definition 8.** *Decision problem  $A = (C, \Sigma, P, u, \pi)$  is more valuable than decision problem  $B = (C', \Sigma, Q, u', \pi')$  if whenever  $g$  is optimal for  $A$  and  $f$  is optimal for  $B$ , we have*

$$\sum_{\omega^* \in S^*} u(g(\omega^*), \omega^*) \pi(\omega^*) \geq \sum_{\omega^* \in S^*} u'(f(\omega^*), \omega^*) \pi'(\omega^*).$$

## 4.1 Results

In the standard framework with partitional information structures, partition  $P^2$  is finer than partition  $P^1$  if and only if it is more valuable for all  $\pi$ ,  $u$  and  $C$  ([Laffont \(1989\)](#)). The following Theorem summarizes the sufficient conditions for information to be valuable in an environment with unawareness.

**Theorem 1.** *Suppose that  $P^2$  is strongly more informed than  $P^1$ . Decision problem  $(C, \Sigma, P^2, u, \pi)$  is more valuable than  $(C, \Sigma, P^1, u, \pi)$  if one of the following is true:*

1.  $(P^2, \pi)$  satisfies conditional independence, or,
2.  $(P^1, \pi)$  satisfies conditional independence and awareness for  $P^2$  is nested and more informative.

*Moreover, whenever one of the properties is violated there exist  $u, C, \pi$  where all other properties are satisfied and  $(C, \Sigma, P^1, u, \pi)$  is strictly more valuable than  $(C, \Sigma, P^2, u, \pi)$ .*

Recall that agent 2 is strongly more informed if whenever agent 1 knows an event  $E$ , then if agent 2 is aware of  $E$  he knows it as well, or, if he is unaware of it, he knows another event  $E'$  that logically implies  $E$ . Hence, a low level of awareness can be compensated by a high level of information. The first part of the Theorem specifies that if the more informed agent 2 does not misuse his awareness signal, information is valuable. If, on the other hand, the less informed agent 1 does not misuse his signal, then information is still valuable provided 2's awareness is nested and more informative.

One reason why the converse can fail to be true is that the payoff relevant state space  $S_0$  may contain too few states. For example, it may be that the only two states that describe different information and awareness for the two agents project to the same payoff relevant state. In that case, differences in information do not imply differences in payoffs. We say that  $S_0$  is non-degenerate for information structure  $P$  if whenever two states are distinguishable according to  $P$ , they can be distinguished also in terms of payoffs.

**Definition 9. [Non-degeneracy]** *The payoff relevant state space  $S_0$  is non-degenerate for  $P$  if for all  $\omega^* \in S^*$ ,  $\omega, \omega' \in P(\omega^*)$  and  $\omega \neq \omega'$  imply  $\omega_{S_0} \neq \omega'_{S_0}$ .*

Under the assumption of non-degeneracy, the following Proposition shows that a necessary condition for an information structure to be more valuable is to be weakly (but not strongly) more informed.

**Proposition 1.** *Suppose that for all  $u, \pi, C$ , decision problem  $(C, \Sigma, P^2, u, \pi)$  is more valuable than  $(C, \Sigma, P^1, u, \pi)$ . If  $S_0$  is non-degenerate for  $P^2$  then  $P^2$  is weakly more informed.*

## 5 Speculation

Speculation is examined in three different settings. Both information processing errors outlined above are relevant here but for different reasons. A low level of awareness is responsible for leading agents to make wrong inferences about the information of others. Hence, existing trading opportunities, if they are beyond one’s awareness, can be left unexplored. Below we provide such an example where although it is common knowledge that there should be no trade, such a trade exists. However, under common priors, common knowledge of trade cannot occur. That is, unaware agents are rational enough to exhaust all trading possibilities that can be described within their common awareness.

Consider the following example, also outlined in Galanis (2007), of two agents with constant but different awareness. The agent with the low level of awareness makes false inferences about the other agent’s information and actions. There are two questions, concerning the possible prices and interest rates, each with two possible answers, “high” and “low”. State space  $S_0 = \{\omega_5, \omega_6\}$  specifies whether prices are high or low, whereas state space  $S^* = \{\omega_1, \omega_2, \omega_4\}$  specifies whether prices and interest rates are high or low. Agent 1 is always aware of both dimensions, and he always gets information about interest rates. Moreover, he knows a theorem that states “low interest rates imply high prices”.<sup>9</sup> Hence, when interest rates are low, he knows that prices are high. Agent 1’s information is as follows.  $P^1(\omega_1) = P^1(\omega_2) = \{\omega_1, \omega_2\}$ ,  $P^1(\omega_4) = \{\omega_4\}$ ,  $P^1(\omega_5) = P^1(\omega_6) = \{\omega_5, \omega_6\}$ . On the other hand, agent 2 is only aware of prices. Since he is unaware of interest rates and of the theorem, he never knows whether prices are high or low. His state space is  $S_0$  and his information is represented by the coarsest partition of  $S_0$ ; for all  $\omega \in \Sigma = S^* \cup S_0$ ,  $P^1(\omega) = S_0$ .

The fact that agent 2’s awareness is constant implies that he does not underestimate or overestimate the occurrence of any event. However, the low level of his awareness leads to wrong reasoning about agent 1’s information. Since he is unaware of the theorem, he falsely concludes that also agent 1 is always unable to know when prices are high or low. In other words, agent 2 thinks that 1’s information is given by  $P^1(\omega_5) = P^1(\omega_6) = S_0$ . The information structure of both agents is depicted in Figure 1.

Let  $t^i(\omega, E)$  be agent  $i$ ’s posterior at  $\omega \in \Sigma$  about event  $E \subseteq \Omega^i(\omega)$ . A probability distribution on the full state space  $S^*$  is a prior for  $i$  if it generates his posteriors.

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<sup>9</sup>Hence, the state  $\omega_3$ , “low interest rates, low prices”, is impossible.

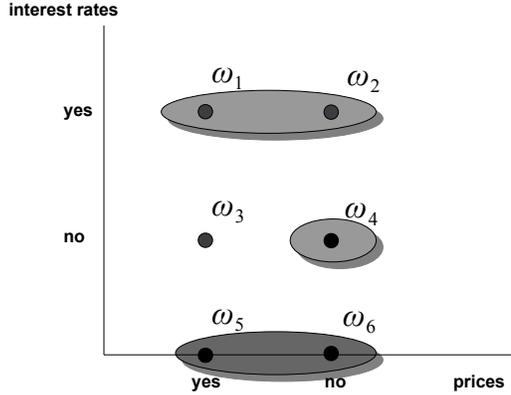


Figure 1: Low level of awareness allows for speculative trade

**Definition 10.** A prior for agent  $i$  is a probability distribution  $\pi \in \Delta S^*$  such that for each  $\omega \in \Sigma$ , if  $\pi(P^i(\omega)^{S^*}) > 0$ , then  $t^i(\omega, E) = \pi|_{\Omega(\omega)}(E|P^i(\omega))$  for each  $E \subseteq \Omega(\omega)$ .

In the example, both agents share a common prior  $\pi$  on  $S^*$ , where  $\pi(\omega_1) = 1/2$ ,  $\pi(\omega_2) = 1/4$ , and  $\pi(\omega_4) = 1/4$ .<sup>10</sup> Since agent 2 is always unaware of  $S^*$ , his posteriors are derived from the marginal of  $\pi$  on  $S_0$ .

A bet is only defined in the common state space  $S_0$  and generates, at each state  $\omega \in S_0$ , gains and losses that add up to zero.

**Definition 11.** A bet  $b$  is a collection of functions  $b^i : S_0 \rightarrow \mathbb{R}$ , one for each agent  $i$ , such that for each  $\omega \in S_0$ ,  $\sum_{i \in I} b^i(\omega) = 0$ . Agent  $i$  expects positive gain from bet  $b$  at  $\omega^* \in S^*$  if  $\sum_{\omega \in \Omega^i(\omega^*)} t^i(\omega^*, \omega) b^i(\omega_{S_0}) > 0$ .

Recall that the payoff relevant state space  $S_0$  is the meet of all state spaces:  $S_0 = \bigwedge \mathcal{S}$ . In this interactive setting, this implies that it is always common knowledge that everyone is aware of  $S_0$  and of any bet defined on  $S_0$ .<sup>11</sup>

Note that at  $\omega_5, \omega_6$  both agents have identical posteriors. Hence, at both  $\omega_5$  and  $\omega_6$  there is no bet from which both agents expect positive gains. Moreover, the event  $S_0 = \{\omega_5, \omega_6\}$  is common knowledge at  $\omega_4$ , which means that it is common knowledge that there is no trade. However, this is true only within the limited state space  $S_0$ . For example, if  $b^1(\omega_5) = -1.1$ ,  $b^1(\omega_6) = 0.9$ ,  $b^2 = -b^1$ , then both agents expect positive gains at  $\omega_4$ . But this reasoning is above agent 2's awareness.

Another observation is that there does not exist a bet that makes both agents expect positive gains always. If such a bet existed, it would specify a positive gain for agent 1 at  $\omega_6$ , because at  $\omega_4$  he assigns probability 1 to  $\omega_6$ . But  $b^1(\omega_6) > 0$  implies  $b^2(\omega_6) < 0$

<sup>10</sup>Since  $\omega_3$  is impossible we do not include it in the state space  $S^*$ .

<sup>11</sup>Formally, for all  $\omega^* \in S^*$ , it is commonly known that everyone is aware of  $S_0$ . For a more detailed exposition of common knowledge of awareness, see Galanis (2007).

and therefore  $b^2(\omega_5) > 0$  and  $b^1(\omega_5) < 0$ . But this means that there is no bet that would make both agents expect positive gains at  $\omega_1$  because agent 1's posteriors are  $2/3$  for  $\omega_1$  and  $1/3$  for  $\omega_2$  while 2's posteriors are  $1/2$  for both  $\omega_5$  and  $\omega_6$ .

Finally, suppose that at  $\omega_4$  agent 1 announces that his posterior about  $\omega_6$  is 1. This is a totally unexpected announcement from the perspective of agent 2, who thinks that the posterior of both agents is  $1/2$ . How will he react? We answer this question in section 6, by constructing a dynamic model of updating awareness, whereas we generalize the observations described above in the following section.

## 5.1 Results

The example above shows that although agents can exhaust trading opportunities that can be described within their common awareness, trading opportunities can still exist. The following result shows that with common priors, there cannot be common knowledge trade. Hence, unaware agents are rational enough to understand that common knowledge of trade exhausts all trading opportunities within their common awareness, just like in the context of the standard model of knowledge with partitioned informational structures.<sup>12</sup>

**Theorem 2.** *If at  $\omega^* \in S^*$  it is common knowledge that there is a bet  $b$  from which all agents expect positive gains, then there is no common prior.*

This is in contrast with models of unawareness (Xiong (2007)) and bounded reasoning (Geanakoplos (1989)) that employ the standard framework, where common knowledge trade is possible. Moreover, if we were to model unawareness of an event by assigning zero probability to that event, then “unaware” agents would engage in common belief trade. Hence, this result provides a behavioral implication which distinguishes the present model from standard state space models of bounded reasoning. Note that no common knowledge trade has also been shown by Heifetz et al. (2009) and by Galanis (2007) (in the form of the agreeing to disagree result).

The next question is whether we can have a trade that always guarantees positive expected gains for everyone. In the context of the standard model where agents make no mistakes, non existence of such a trade is characterized by the existence of a common prior. Since anything that is always true is always common knowledge, common priors are equivalent to not having a bet for which it is always common knowledge that it is beneficial in expectation for everyone.

If we allow for unawareness, then this result is false. First, what is always true is not always common knowledge. The reason is that if one is not fully aware, he may fail to realize that something is always true.<sup>13</sup> Second, as in the value of information problem, agents may overestimate or underestimate the occurrence of events depending on how their awareness varies. The following result shows that if the priors of all agents satisfy conditional independence, then they can trade only if they have different priors.

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<sup>12</sup>An example of common knowledge trade with common priors but non partitioned information structures is provided in the appendix.

<sup>13</sup>In the example, agent 2 is unaware (hence he does not know) that agent 1 always knows whether interest rates are high or low.

Third, the converse does not hold. Even in the presence of conditional independence, different priors do not imply trade.

**Theorem 3.** *Suppose that the priors of all agents satisfy conditional independence and there is a bet  $b$  for which all agents always expect a positive gain. Then, there is no common prior. However, no common prior and conditional independence do not imply the existence of such a bet  $b$ .*

The third setting is trade in equilibrium. For that, we need to define the notion of a Bayesian Nash equilibrium in an environment with unawareness. Let  $I$  be a finite set of agents, each having a finite set of actions  $C^i$  and let  $C = \times_{i \in I} C^i$  denote the set of all action profiles. In the single agent case we defined a decision function to be a mapping from the full state space  $S^*$  to the set of actions  $C^i$ . For the multi agent case we generalize by defining the domain of the decision function to be the union of all state spaces  $\Sigma$ .

**Definition 12.** *A Bayesian game with unawareness is a tuple  $(I, C, \Sigma, (P^i)_{i \in I}, (t^i)_{i \in I}, (u^i)_{i \in I})$ , where  $C$  is the set of all action profiles,  $\Sigma$  is the union of all state spaces,  $P^i$  denotes agent  $i$ 's possibility correspondence,  $u^i : C \times \Sigma \rightarrow \mathbb{R}$  denotes his utility function and  $t^i$  his type mapping.*

**Definition 13.** *A strategy for player  $i \in I$  is a function  $f^i : \Sigma \rightarrow C^i$  such that for all  $\omega, \omega' \in \Sigma$ ,  $t^i(\omega, \cdot) = t^i(\omega', \cdot)$  implies  $f^i(\omega) = f^i(\omega')$ .*

**Definition 14.** *Strategies  $(f^i)_{i \in I}$  constitute a Bayesian Nash equilibrium if for all  $\omega \in \Sigma$ , all  $i \in I$  and all  $c^i \in C^i$ ,*

$$\sum_{\omega' \in \Omega^i(\omega)} u^i(f^i(\omega'), f^{-i}(\omega'), \omega') t^i(\omega, \omega') \geq \sum_{\omega' \in \Omega^i(\omega)} u^i(c^i, f^{-i}(\omega'), \omega') t^i(\omega, \omega').$$

In a Bayesian game with unawareness agents have to reason about the actions of others, when choosing their own best response. If the level of their awareness is too low, then they may reason incorrectly about the information and actions of others and as a result have a wrong perception of their payoffs. In the example, when  $\omega_4$  occurs agent 2 incorrectly thinks that agent 1's posterior about  $\omega_6$  is 1. This wrong perception can lead the agent to choose suboptimally in a game. The following condition specifies that an agent's level of awareness does not influence his perception of his payoffs.

**Definition 15** (Projections Preserve Own Payoffs). *A strategy profile  $\{f^i\}_{i \in I}$  satisfies PPOP if for all  $i \in I$ , for all  $\omega \in \Sigma$ ,  $u^i(c, f^{-i}(\omega), \omega) = u^i(c, f^{-i}(\omega_{S_0}), \omega_{S_0})$ , for all  $c \in C$ .*

When all equilibria in a game  $G$  satisfy PPOP we say that  $G$  satisfies PPOP. Agent  $i$ 's ex ante expected utility from strategy profile  $\{f^j\}_{j \in I}$  is

$$U^i(\{f^j\}_{j \in I}) = \sum_{\omega^* \in S^*} \pi^i(\omega^*) u^i(f(\omega^*), \omega^*).$$

Suppose that the agents have arrived at an ex ante Pareto optimal allocation. This means that each agent can stick to his allocation and can guarantee for himself an ex ante payoff of  $\bar{u}^i$ , irrespective of what everyone else is doing. Moreover, there is no strategy profile that ex ante can make everyone weakly better off and at least one strictly better off. Suppose that the agents receive differential information and awareness. Will they be willing to trade? As the following theorem shows, the answer is negative, as long as agents do not misperceive their payoffs due to their low awareness, and do misuse their awareness signal.

**Theorem 4.** *Let  $G$  be a Bayesian game with unawareness that satisfies PPOP and suppose that each agent satisfies either nested awareness or conditional independence. Suppose that each agent  $i$  has an action  $z^i \in C^i$  such that for all  $\{f^j\}_{j \in I}$ ,  $U^i([z^i], \{f^j\}_{j \in I-i}) = \bar{u}^i$ . Moreover, suppose that for all  $\{f^j\}_{j \in I}$ , if  $U^i(\{f^j\}_{j \in I}) \geq \bar{u}^i$  for all  $i \in I$ , then  $f^j(\omega^*) = [z^j]$  for all  $\omega^* \in S^*$ , for all  $j \in I$ . Then,  $G$  has a unique equilibrium such that  $f^i(\omega^*) = [z^i]$  for all  $\omega^* \in S^*$ .*

## 6 Updating awareness

In the example of the previous section the fully aware agent announces an action (bet on the price being high) that the less aware agent cannot rationalize. As a result, the second agent realizes that he is missing or he is unaware of “something”. How will he react to this realization? Will he update his awareness until he can rationalize the announcement? Do announcements reveal awareness, together with information? Will the agents eventually agree on the action to be taken? Finally, how much they need to update and towards which direction?

We answer these questions by constructing a dynamic version of the static model. The setting is similar to that of [Geanakoplos and Polemarchakis \(1982\)](#) where two agents,  $\alpha$  and  $\beta$ , take turns in announcing their posteriors about an event. An announcement by agent  $\alpha$  transmits public information that enables  $\beta$  to update his own information and vice versa. Geanakoplos and Polemarchakis show that if the partition cells are finite, agents will agree on the posteriors after finite steps.

The setting in this paper is different in that agents update their awareness, not only their information. The assumption is that agent  $\alpha$  updates his awareness whenever he hears an announcement that he cannot rationalize. This happens if none of the states he considers possible describe that  $\beta$  would make such an announcement. In other words, updating with the information revealed by the announcement yields the empty set. We interpret this as the realization on the part of the agent that he is missing something. His reaction is to increase his awareness until he can rationalize the announcement. There are many ways in which  $\alpha$ 's awareness can increase. The only requirement we impose is that the increase is minimal.

### 6.1 Updating process

Consider two agents,  $\alpha$  and  $\beta$ , with possibility correspondences  $P^\alpha$  and  $P^\beta$ , respectively. Recall that  $S_0 = \bigwedge S$  is the least expressive state space and that both agents

are aware of it. Consider the event  $A \subseteq S_0$ . At each period  $t$ , both agents take turns in announcing their posteriors about  $A$ .

At each period  $t$  a static model is defined where  $P_t^i$  is  $i$ 's possibility correspondence and where full state  $\omega^*$  occurs. Set  $P_1^\alpha = P^\alpha$ ,  $P_0^\beta = P^\beta$ . At  $t = 1$ ,  $\alpha$  announces

$$q_1^\alpha(\omega^*) = \frac{\pi(P_1^\alpha(\omega^*) \cap A_{\Omega_1^\alpha(\omega^*)})}{\pi(P_1^\alpha(\omega^*))}.$$

This announcement reveals some public information. This information consists of the states that would enable  $\alpha$  to make such an announcement. Because agents have different awareness, they perceive this public information differently. More specifically, if an agent's state space is  $S$ , then he would perceive this public information as the set

$$\alpha_1(S) = \left\{ \omega \in S : q_1^\alpha(\omega^*) = \frac{\pi(P_1^\alpha(\omega) \cap A_{\Omega_1^\alpha(\omega)})}{\pi(P_1^\alpha(\omega))} \right\}.$$

Reacting to the announcement, agent  $\beta$  updates his private information and awareness. We only need to specify the updating process for states in  $\alpha_1(S)$ , because only these states are relevant after the announcement.<sup>14</sup>

Define, for all  $\omega \in \alpha_1(S)$ ,  $S \in \mathcal{S}$ ,

1. If  $P_0^\beta(\omega) \cap \alpha_1(\Omega_0^\beta(\omega)) \neq \emptyset$  then  $P_1^\beta(\omega) = P_0^\beta(\omega) \cap \alpha_1(\Omega_0^\beta(\omega))$ .

This is the standard case of updating information only. If  $\omega$  was the true state, then  $P_0^\beta(\omega)$  would be  $\beta$ 's information. After hearing the announcement, he would update by incorporating set  $\alpha_1(\Omega_0^\beta(\omega))$ , the public information revealed by the announcement. As long as the intersection of these two sets is nonempty, agent  $\beta$  has no reason to suspect that he is missing "something".

2. If  $P_0^\beta(\omega) \cap \alpha_1(\Omega_0^\beta(\omega)) = \emptyset$  then  $P_1^\beta(\omega) = (P_0^\beta(\omega))^{S'} \cap \alpha_1(S') \neq \emptyset$  for some minimal  $S'$  where  $\Omega_0^\beta(\omega) \preceq S' \preceq S$ .<sup>15</sup>

This is the case where agent  $\beta$  realizes at  $\omega$  that he is missing something. His reaction is to update his awareness so that he can rationalize  $\alpha$ 's announcement. If he updates to state space  $S'$ , then his new information is the enlargement of his old information to the new state space  $S'$ , intersected with the public information perceived from state space  $S'$ ,  $\alpha_1(S')$ . As long as this set is not empty, it rationalizes the announcement. Note that that for  $S' = S$  and by Generalized Reflexivity (nondelusion),  $(P_0^\beta(\omega))^{S'} \cap \alpha_1(S') \neq \emptyset$ . Hence, such a  $S'$  always exists.

Agent  $\beta$  announces

$$q_1^\beta(\omega^*) = \frac{\pi(P_1^\beta(\omega^*) \cap A_{\Omega_1^\beta(\omega^*)})}{\pi(P_1^\beta(\omega^*))}$$

and the public information revealed, described in state space  $S$ , is given by

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<sup>14</sup>For  $\omega \in S \setminus \alpha_1(S)$ ,  $P_1^\beta(\omega)$  is defined in an arbitrary way, with the only requirement that  $P_1^\beta$  satisfies the five properties outlined in section 3.

<sup>15</sup> $S'$  is minimal in the sense that there does not exist  $S'' \prec S'$  such that  $(P_0^\beta(\omega))_{S''} \cap \alpha_1(S'') \neq \emptyset$ . There may exist many such minimal state spaces.

$$\beta_1(S) = \left\{ \omega \in \alpha_1(S) : q_1^\beta(\omega^*) = \frac{\pi(P_1^\beta(\omega) \cap A_{\Omega_1^\beta(\omega)})}{\pi(P_1^\beta(\omega))} \right\}.$$

At time  $t$ , agent  $\alpha$  updates as follows. For all  $\omega \in \beta_{t-1}(S)$ ,  $S \in \mathcal{S}$ ,

1. If  $P_{t-1}^\alpha(\omega) \cap \beta_{t-1}(\Omega_{t-1}^\alpha(\omega)) \neq \emptyset$  then  $P_t^\alpha(\omega) = P_{t-1}^\alpha(\omega) \cap \beta_{t-1}(\Omega_{t-1}^\alpha(\omega))$ .
2. If  $P_{t-1}^\alpha(\omega) \cap \beta_{t-1}(\Omega_{t-1}^\alpha(\omega)) = \emptyset$  then  $P_t^\alpha(\omega) = (P_{t-1}^\alpha(\omega))^{S'} \cap \beta_{t-1}(S') \neq \emptyset$  for some minimal  $S'$  such that  $\Omega_{t-1}^\alpha(\omega) \preceq S' \preceq S$ .

He announces

$$q_t^\alpha(\omega^*) = \frac{\pi(P_t^\alpha(\omega^*) \cap A_{\Omega_t^\alpha(\omega^*)})}{\pi(P_t^\alpha(\omega^*))}$$

and the public information revealed is

$$\alpha_t(S) = \left\{ \omega \in \beta_{t-1}(S) : q_t^\alpha(\omega^*) = \frac{\pi(P_t^\alpha(\omega) \cap A_{\Omega_t^\alpha(\omega)})}{\pi(P_t^\alpha(\omega))} \right\}.$$

For all  $\omega \in \alpha_t(S)$ ,  $S \in \mathcal{S}$ ,  $\beta$  updates as follows.

1. If  $P_{t-1}^\beta(\omega) \cap \alpha_t(\Omega_{t-1}^\beta(\omega)) \neq \emptyset$  then  $P_t^\beta(\omega) = P_{t-1}^\beta(\omega) \cap \alpha_t(\Omega_{t-1}^\beta(\omega))$ .
2. If  $P_{t-1}^\beta(\omega) \cap \alpha_t(\Omega_{t-1}^\beta(\omega)) = \emptyset$  then  $P_t^\beta(\omega) = (P_{t-1}^\beta(\omega))^{S'} \cap \alpha_t(S') \neq \emptyset$  for some minimal  $S'$  such that  $\Omega_{t-1}^\beta(\omega) \preceq S' \preceq S$ .

Agent  $\beta$  announces

$$q_t^\beta(\omega^*) = \frac{\pi(P_t^\beta(\omega^*) \cap A_{\Omega_t^\beta(\omega^*)})}{\pi(P_t^\beta(\omega^*))}$$

and the public information revealed is

$$\beta_t(S) = \left\{ \omega \in \alpha_t(S) : q_t^\beta(\omega^*) = \frac{\pi(P_t^\beta(\omega) \cap A_{\Omega_t^\beta(\omega)})}{\pi(P_t^\beta(\omega))} \right\}.$$

In general,  $\alpha_t(S)$  or  $\beta_t(S)$  can be empty. In the example, when the fully aware agent announces a posterior of 1 for the event “prices are high”,  $\alpha_1(S_0)$  is empty. If  $\alpha_t(S)$  is empty then no updating occurs at any state in  $S$  and  $\alpha_k(S)$  remains empty for all  $k \geq t$ .

Finally, we require that at every time  $t$ ,  $P_t^\alpha$  and  $P_t^\beta$  satisfy all properties outlined in section 3. It is not true in general that for any given  $P^\alpha$  and  $P^\beta$ , the updating process yields at each  $t$  a pair  $P_t^\alpha, P_t^\beta$  where the properties are satisfied. The reason is that although the true state  $\omega^*$  belongs to  $\alpha_t(S^*)$  and  $\beta_t(S^*)$ , this is not necessarily true for  $\alpha_t(S), \beta_t(S)$  where  $S \prec S^*$ . Hence, a less than fully aware agent may have a wrong view of the public information revealed and hence update by excluding the true state. But then, Generalized Reflexivity is not satisfied. For agreement we need that unaware agents do not have a wrong view of the information revealed by the announcement.

## 6.2 Reaching an agreement

The updating process described above specifies that the agents increase their awareness when they hear an announcement that contradicts their knowledge. A crucial feature of the process is that they are able to update as much as necessary (but no more than that) in order to rationalize the other agent's announcement. The following Theorem shows that agents reach an agreement eventually.

**Theorem 5.** *There exists a finite  $t$  such that  $q_t^\alpha(\omega^*) = q_t^\beta(\omega^*)$ . Moreover, agent  $i$  updates his awareness at  $t$  if and only if  $\Omega_{t-1}^i(\omega^*) \wedge \Omega_{t-1}^j(\omega^*) < \Omega_t^i(\omega^*) \wedge \Omega_{t-1}^j(\omega^*)$ , where  $i, j = \alpha, \beta$ . Finally, suppose that whenever  $S' \succeq S$  and  $\omega_{S'}^* \in i_t(S')$ , we have  $\omega_S^* \in i_t(S)$ , where  $i = \alpha, \beta$ . Then, for each  $t > 1$ ,  $\Omega_t^\alpha(\omega^*) \vee \Omega_t^\beta(\omega^*) = \Omega_{t-1}^\alpha(\omega^*) \vee \Omega_{t-1}^\beta(\omega^*)$ .*

The two other results describe when and how agents update their awareness. The first shows that an agent updates his awareness if and only if he becomes aware of something that the other agent was already aware of. This is the *only* way of rationalizing the announcement. Effectively, this means that a necessary condition for agreement is not that agents are really smart and can always increase their awareness, but that they can work out what is going on inside the other agent's head. By putting together two really smart people with totally different backgrounds it is not guaranteed that they will eventually agree. Being able to understand how the other one thinks is more important than having the means to increase your awareness in arbitrary directions. One could say that agreement is more the result of similar backgrounds, than unbounded rationality.

Whereas the previous result places a lower bound on the updating of awareness, the last part specifies an upper bound. Suppose that more expressive state spaces are always better at describing the agents' announcements. That is, whenever the truth ( $\omega^*$ ) belongs to the public information described by  $S'$ , and  $S$  is more expressive, then the truth also belongs to the public information described by  $S$ . This property is always true when  $S = S^*$ , the full state space, but may fail otherwise. When the property is true the "sum" (join) of the agents' awareness is constant across time. Hence, whenever one updates his awareness, at most he can be aware of everything that the other agent is aware of.

## A Appendix

*Proof of first part of Theorem 1.* Suppose  $f$  is optimal for  $(C, \Sigma, P^1, u, \pi)$  and  $g$  is optimal for  $(C, \Sigma, P^2, u, \pi)$ . Fix  $\omega^* \in S^*$  with  $\pi(\omega^*) > 0$ . For all  $c \in C$ , we have

$$\sum_{\omega \in P^2(\omega^*)} u(g(\omega^*), \omega) \pi|_{\Omega^2(\omega^*)}(\omega) \geq \sum_{\omega \in P^2(\omega^*)} u(c, \omega) \pi|_{\Omega^2(\omega^*)}(\omega).$$

Conditional independence and generalized reflexivity imply that for  $\omega, \omega' \in P^2(\omega^*)$  such that  $\pi|_{\Omega^2(\omega^*)}(\omega), \pi|_{\Omega^2(\omega^*)}(\omega') > 0$ , we have

$$\frac{\pi(\mathcal{E}^2(\omega^*) \cap \omega^{S^*})}{\pi|_{\Omega^2(\omega^*)}(\omega)} = \frac{\pi(\mathcal{E}^2(\omega^*) \cap \omega'_{S^*})}{\pi|_{\Omega^2(\omega^*)}(\omega')} > 0.$$

Multiplying by that number we have that for all  $c \in C$ ,

$$\begin{aligned} \sum_{\omega \in P^2(\omega^*)} u(g(\omega^*), \omega) \pi(\mathcal{E}^2(\omega^*) \cap \omega^{S^*}) &\geq \sum_{\omega \in P^2(\omega^*)} u(c, \omega) \pi(\mathcal{E}^2(\omega^*) \cap \omega^{S^*}) \implies \\ \sum_{\omega \in P^2(\omega^*)} u(g(\omega^*), \omega) \sum_{\omega_1^* \in \mathcal{E}^2(\omega^*) \cap \omega^{S^*}} \pi(\omega_1^*) &\geq \sum_{\omega \in P^2(\omega^*)} u(c, \omega) \sum_{\omega_1^* \in \mathcal{E}^2(\omega^*) \cap \omega^{S^*}} \pi(\omega_1^*). \end{aligned}$$

Since  $\{\omega_1^*\}_{S_0} = \omega_{S_0}$  for all  $\omega_1^* \in \omega^{S^*}$  we have

$$\begin{aligned} \sum_{\omega \in P^2(\omega^*)} \sum_{\omega_1^* \in \mathcal{E}^2(\omega^*) \cap \omega^{S^*}} u(g(\omega^*), \omega_1^*) \pi(\omega_1^*) &\geq \sum_{\omega \in P^2(\omega^*)} \sum_{\omega_1^* \in \mathcal{E}^2(\omega^*) \cap \omega^{S^*}} u(c, \omega_1^*) \pi(\omega_1^*) \implies \\ \sum_{\omega_1^* \in (P^2(\omega^*))_{S^*} \cap \mathcal{E}^2(\omega^*)} u(g(\omega^*), \omega_1^*) \pi(\omega_1^*) &\geq \sum_{\omega_1^* \in (P^2(\omega^*))_{S^*} \cap \mathcal{E}^2(\omega^*)} u(c, \omega_1^*) \pi(\omega_1^*). \end{aligned}$$

Hence, conditional independence implies that for any  $\omega^* \in S^*$ , the agent's best action at  $P^2(\omega^*)$  is also the best action at  $(P^2(\omega^*))_{S^*} \cap \mathcal{E}^2(\omega^*)$ . Next, we show that the full state space  $S^*$  is partitioned by

$$\{(P^2(\omega^*))_{S^*} \cap \mathcal{E}^2(\omega^*)\}_{\omega^* \in S^*}.$$

Suppose  $\omega_1^* \in (P^2(\omega^*))_{S^*} \cap \mathcal{E}^2(\omega^*)$ . Then,  $\Omega^2(\omega^*) = \Omega^2(\omega_1^*)$  and  $\{\omega_1^*\}_{\Omega^2(\omega^*)} \in P^2(\omega^*)$ . Generalized Reflexivity implies  $\{\omega_1^*\}_{\Omega^2(\omega^*)} \in P^2(\omega_1^*)$  and Stationarity implies  $P^2(\omega_1^*) = P^2(\{\omega_1^*\}_{\Omega^2(\omega^*)}) = P^2(\omega^*)$ . Hence,  $(P^2(\omega_1^*))_{S^*} \cap \mathcal{E}^2(\omega_1^*) = (P^2(\omega^*))_{S^*} \cap \mathcal{E}^2(\omega^*)$ .

We also show that  $\omega_1^*, \omega_2^* \in (P^2(\omega^*))_{S^*} \cap \mathcal{E}^2(\omega^*)$  implies  $g(\omega_1^*) = g(\omega_2^*)$ . Since  $\Omega^2(\omega_1^*) = \Omega^2(\omega_2^*) = \Omega^2(\omega^*)$ , Stationarity implies that either  $P^2(\omega_1^*) = P^2(\omega^*)$  or  $P^2(\omega_1^*) \cap P^2(\omega^*) = \emptyset$ . Generalized Reflexivity and  $\omega_1^* \in (P^2(\omega^*))_{S^*}$  imply  $P^2(\omega_1^*) = P^2(\omega^*)$ . Similarly for  $\omega_2^*$ , so we have  $P^2(\omega_1^*) = P^2(\omega_2^*)$ . Hence,  $g(\omega_1^*) = g(\omega_2^*)$ . The same argument shows that  $\omega_1^*, \omega_2^* \in (P^1(\omega^*))_{S^*} \cap \mathcal{E}^1(\omega^*)$  implies  $f(\omega_1^*) = f(\omega_2^*)$ .

We showed that for each element of the partition  $\{(P^2(\omega^*))_{S^*} \cap \mathcal{E}^2(\omega^*)\}_{\omega^* \in S^*}$ , agent 2 picks an action that maximizes his expected utility. Moreover, agent 1 picks one action for each element of the partition,  $\{(P^1(\omega^*))_{S^*} \cap \mathcal{E}^1(\omega^*)\}_{\omega^* \in S^*}$ . This action may not necessarily be optimal.

Finally, we need to show that the former partition is finer than the latter if and only if agent 2 is strongly more informed. One direction is obvious, so for the other direction suppose  $\omega_1^* \in (P^2(\omega^*))_{S^*} \cap \mathcal{E}^2(\omega^*)$ . Then,  $\omega_1^* \in (P^2(\omega^*))_{S^*}$  and Stationarity, together with PPI, implies  $\Omega^1(\omega_1^*) \succeq \Omega^1(\omega^*)$ . Suppose  $\Omega^1(\omega_1^*) \succ \Omega^1(\omega^*)$ . Then,  $\omega^* \notin (P^1(\omega_1^*))_{S^*}$ , which implies  $\omega^* \notin (P^2(\omega_1^*))_{S^*}$ . But this contradicts the fact that  $\{(P^2(\omega^*))_{S^*} \cap \mathcal{E}^2(\omega^*)\}_{\omega^* \in S^*}$  is a partition. Hence,  $\Omega^1(\omega_1^*) = \Omega^1(\omega^*)$  and  $\omega_1^* \in (P^1(\omega^*))_{S^*} \cap \mathcal{E}^1(\omega^*)$ . Note that if the agent is always more aware and weakly more informed, then he is strongly more informed.

The main example shows that if conditional independence is violated but the agent is strongly more informed, then he may be worse off. In Example 1, the agent satisfies conditional independence, he is weakly (but not strongly) more informed and he is strictly worse off.

□

The proof of the second part of Theorem 1 is given in three steps. First, using  $P^2$  we define a possibility correspondence  $P : S^* \rightarrow 2^{S^*}$  and show that it satisfies non-delusion, KTYK and nested, three properties which are discussed in Geanakoplos (1989). Then, we show that if  $g$  is optimal for  $P^2$  then it is also optimal for  $P$  in a suitably defined problem in the Geanakoplos' setting. Finally, we apply Theorem 1 in Geanakoplos (1989).

Define the mapping  $P : S^* \rightarrow 2^{S^*}$  such that for all  $\omega^* \in S^*$ ,  $P(\omega^*) = (P^2(\omega^*))_{S^*}$ . We show that  $P$  also satisfies the following three properties, discussed in Geanakoplos (1989):

**Definition 16.**

- **Non-Delusion** For all  $\omega^* \in S^*$ ,  $\omega^* \in P(\omega^*)$ .
- **KTYK**  $\omega_1^* \in P(\omega^*) \implies P(\omega_1^*) \subseteq P(\omega^*)$ .
- **Nested** For any  $\omega^*, \omega_1^* \in S^*$ , either  $P(\omega^*) \cap P(\omega_1^*) = \emptyset$ , or  $P(\omega^*) \subseteq P(\omega_1^*)$  or  $P(\omega_1^*) \subseteq P(\omega^*)$ .

**Lemma 1.** *Suppose that  $P^2$  satisfies nested awareness. Then, the possibility correspondence  $P$  satisfies non-delusion, KTYK and nested.*

*Proof.*  $P$  satisfies non-delusion because  $P^2$  satisfies Generalized Reflexivity. For KTYK, suppose  $\omega_1^* \in P(\omega^*) = (P^2(\omega^*))_{S^*}$ . Then,  $\{\omega_1^*\}_{\Omega^2(\omega^*)} \in P^2(\omega^*)$ . Stationarity implies  $P^2(\{\omega_1^*\}_{\Omega^2(\omega^*)}) = P^2(\omega^*)$ . PPI implies  $(P^2(\omega_1^*))_{S^*} \subseteq (P^2(\{\omega_1^*\}_{\Omega^2(\omega^*)}))_{S^*} = (P^2(\omega^*))_{S^*}$ . Hence  $P(\omega_1^*) \subseteq P(\omega^*)$ .

For nested, suppose  $P(\omega^*) \cap P(\omega_1^*) \neq \emptyset$ . Take  $\omega_2^* \in P(\omega^*) \cap P(\omega_1^*)$ . Then  $\omega_2^* \in (P^2(\omega^*))_{S^*} \cap (P^2(\omega_1^*))_{S^*}$ . As in the previous paragraph, this implies that  $P^2(\{\omega_2^*\}_{\Omega^2(\omega_1^*)}) = P^2(\omega_1^*)$  and  $P^2(\{\omega_2^*\}_{\Omega^2(\omega^*)}) = P^2(\omega^*)$ . Without loss of generality, suppose that  $\Omega^2(\omega^*) \preceq \Omega^2(\omega_1^*)$ . PPI implies that  $(P^2(\{\omega_2^*\}_{\Omega^2(\omega_1^*)}))_{S^*} \subseteq (P^2(\{\omega_2^*\}_{\Omega^2(\omega^*)}))_{S^*}$  and hence,  $P(\omega_1^*) \subseteq P(\omega^*)$ .  $\square$

The setting in Geanakoplos (1989) specifies a finite state space  $S^*$ , a possibility correspondence  $P : S^* \rightarrow 2^{S^*}$  and an action set  $C$ . Let  $u : C \times S^* \rightarrow \mathbb{R}$  and let  $\pi$  be a measure on  $S^*$ .

**Definition 17.** *In the Geanakoplos' setting, a decision function  $f : S^* \rightarrow C$  is optimal for the decision problem  $(C, S^*, P, u, \pi)$  iff*

- $P(\omega^*) = P(\omega_1^*) \implies f(\omega^*) = f(\omega_1^*)$
- For all  $\omega^* \in S^*$  and  $c \in C$ ,

$$\sum_{\omega_1^* \in P(\omega^*)} u(f(\omega^*), \omega_1^*) \pi(\omega_1^*) \geq \sum_{\omega_1^* \in P(\omega^*)} u(c, \omega_1^*) \pi(\omega_1^*)$$

Given a decision problem  $(C, \Sigma, P^2, u, \pi)$  we define the decision problem  $(C, S^*, P, u', \pi)$  in the setting of Geanakoplos (1989). The possibility correspondence  $P : S^* \rightarrow 2^{S^*}$  is such that for all  $\omega^* \in S^*$ ,  $P(\omega^*) = (P^2(\omega^*))_{S^*}$ . The utility function  $u' : C \times S^* \rightarrow \mathbb{R}$  is defined such that  $u'(c, \omega^*) = u(c, \omega_{S_0}^*)$  for all  $\omega^* \in S^*$ , for all  $c \in C$ .

**Lemma 2.** *Decision function  $g$  is optimal for  $(C, \Sigma, P^2, u, \pi)$  if and only if  $g$  is optimal for  $(C, S^*, P, u', \pi)$ .*

*Proof.* First we show that for any  $\omega^*, \omega_1^* \in S^*$ ,  $P^2(\omega^*) = P^2(\omega_1^*) \iff (P^2(\omega^*))_{S^*} = (P^2(\omega_1^*))_{S^*}$ . One direction is obvious so for the other suppose that  $(P^2(\omega^*))_{S^*} = (P^2(\omega_1^*))_{S^*}$ . Since  $\omega^* \in (P^2(\omega_1^*))_{S^*}$  we have  $\omega_{\Omega^2(\omega_1^*)}^* \in P^2(\omega_1^*)$ , which implies  $P^2(\omega_{\Omega^2(\omega_1^*)}^*) = P^2(\omega_1^*)$ . PPI implies  $P^2(\omega^*)^\uparrow \subseteq P^2(\omega_{\Omega^2(\omega_1^*)}^*)^\uparrow = P^2(\omega_1^*)^\uparrow$ . Similarly, since  $\omega_1^* \in (P^2(\omega^*))_{S^*}$ , we have  $P^2(\omega^*)^\uparrow = P^2(\omega_1^*)^\uparrow$ , which implies  $P^2(\omega^*) = P^2(\omega_1^*)$ .

If  $g$  is optimal for the first problem, then for any  $\omega^* \in S^*$  and any  $c \in C$ ,

$$\sum_{\omega \in P^2(\omega^*)} u(g(\omega^*), \omega_{S_0}) \pi(\omega^{S^*}) \geq \sum_{\omega \in P^2(\omega^*)} u(c, \omega_{S_0}) \pi(\omega^{S^*}).$$

Fix  $\omega^* \in S^*$  and take  $\omega \in P^2(\omega^*)$ . For any  $\omega_1^* \in \omega^{S^*}$ , we have that  $u'(g(\omega^*), \omega_1^*) = u(g(\omega^*), \{\omega_1^*\}_{S_0}) = u(g(\omega^*), \omega_{S_0})$ . We also have that  $\pi(\omega^{S^*}) = \sum_{\omega_1^* \in \omega^{S^*}} \pi(\omega_1^*)$ . Combin-

ing these two we have

$$\sum_{\omega \in P^2(\omega^*)} \sum_{\omega_1^* \in \omega^{S^*}} u'(g(\omega^*), \omega_1^*) \pi(\omega_1^*) \geq \sum_{\omega \in P^2(\omega^*)} \sum_{\omega_1^* \in \omega^{S^*}} u'(c, \omega_1^*) \pi(\omega_1^*),$$

which is equivalent to

$$\sum_{\omega_1^* \in (P^2(\omega^*))_{S^*}} u'(g(\omega^*), \omega_1^*) \pi(\omega_1^*) \geq \sum_{\omega_1^* \in (P^2(\omega^*))_{S^*}} u'(c, \omega_1^*) \pi(\omega_1^*).$$

Since  $P(\omega^*) = (P^2(\omega^*))_{S^*}$ ,  $g$  is optimal for the second problem in the Geanakoplos' setting. The other direction is similar.  $\square$

The following theorem is proved in [Geanakoplos \(1989\)](#).

**Theorem 6** ([Geanakoplos \(1989\)](#)). *Let  $P$  satisfy non-delusion, nested and KTYK. Let  $R$  be a partition of  $S^*$  that is a coarsening of  $P$ . Let  $g, f$  be optimal for  $(C, S^*, P, u', \pi)$  and  $(C, S^*, R, u', \pi)$  respectively. Then,*

$$\sum_{\omega^* \in S^*} u'(g(\omega^*), \omega^*) \pi(\omega^*) \geq \sum_{\omega^* \in S^*} u'(f(\omega^*), \omega^*) \pi(\omega^*).$$

*Proof of the second part of Theorem 1.* Suppose  $f$  is optimal for  $(C, \Sigma, P^1, u, \pi)$  and  $g$  is optimal for  $(C, \Sigma, P^2, u, \pi)$ . Define  $Q : S^* \rightarrow 2^{S^*}$  such that  $Q(\omega^*) = P^1(\omega^*)^{S^*} \cap \mathcal{E}^1(\omega^*)$ . Since  $P^1$  satisfies conditional independence, we know from the proof of the first part of Theorem 1 that  $Q$  partitions the full state space and that  $f$  is optimal for  $(C, S^*, Q, u', \pi)$ . Define  $P : S^* \rightarrow 2^{S^*}$  such that for all  $\omega^* \in S^*$ ,  $P(\omega^*) = (P^2(\omega^*))_{S^*}$ . By Lemma 1,  $P$  satisfies non-delusion, nested and KTYK. Since  $\mathcal{E}^2(\omega^*) \subseteq \mathcal{E}^1(\omega^*)$  for all  $\omega^* \in S^*$  and  $P^2$  is strongly more informed than  $P^1$ , we have that  $P$  is finer than  $Q$ . By applying Lemma 2,  $g$  is optimal for  $(C, S^*, P, u', \pi)$ . By Theorem 6,  $g$  attains a higher ex ante expected utility than what  $f$  attains.

The main example shows that if the less informed agent does not satisfy conditional independence while the awareness of the strongly more informed agent is nested and

more informative, then he may be worse off. In the appendix and with examples 1, 2 and 3 we show that agent 2 can be worse off if, respectively, strongly more informed, nested awareness, informed awareness fail but all other properties hold.  $\square$

*Proof of Proposition 1.* Suppose that  $P^2$  is not weakly more informed than  $P^1$ . Then, there exist  $\omega^*, \omega_1$  such that  $\omega_1 \in P^2(\omega^*)_S$ ,  $\omega_1 \notin P^1(\omega^*)_S$ , where  $S = \Omega^1(\omega^*) \wedge \Omega^2(\omega^*)$ . Let  $\omega_2 \in P^2(\omega^*)$  be such that  $\omega_{2S} = \omega_1$ . By Generalized Reflexivity and since  $\omega_1 \notin P^1(\omega^*)_S$ , we have that  $\omega_1 \neq \omega_S^*$ , which implies that  $\omega_2 \neq \omega_{\Omega^2(\omega^*)}^*$ . By non-degeneracy of  $P^2$ ,  $\omega_{2S_0} \neq \omega_{S_0}^*$ . By Generalized Reflexivity,  $\omega_S^* \in P^2(\omega^*)_S, P^1(\omega^*)_S$ . Let  $C = \{c_1, c_2\}$  and consider the following payoffs:  $u(c_1, \omega_{S_0}^*) = -1$ ,  $u(c_1, \omega_{2S_0}) = 1.1$ ,  $u(c_2, \omega_{S_0}^*) = 8$ ,  $u(c_2, \omega_{2S_0}) = -8$ .

Let  $\pi(\omega^*) = 1/2$  and  $\pi(\omega_2^*) = 1/2$ , where  $\omega_{2\Omega^2(\omega^*)}^* = \omega_2$ . At  $\omega^*$ , 2's optimal action is  $c_1$ , while 1's optimal action is  $c_2$ . At  $\omega_2^*$ , from Generalized Reflexivity, both agents assign probability at least 1/2 to state  $\omega_{2S}$  and their optimal action is  $c_1$ . Hence, the decision problem  $(C, \Sigma, P^1, u, \pi)$  is more valuable than  $(C, \Sigma, P^2, u, \pi)$ , a contradiction.  $\square$

*Proof of Theorem 2.* Suppose there is a common prior  $\pi$ , there are bets  $b^i : S_0 \rightarrow \mathbb{R}$ ,  $i \in I$ , and an event  $E^*$  such that  $S_0 \preceq E^*$ , and for each  $\omega \in E^*$ , all agents expect positive gains from their respective bets. Suppose that at  $\omega^* \in S^*$ , event  $E^*$  is common knowledge and  $E^* \subseteq \Omega^\wedge(\omega^*)$ , where  $\Omega^\wedge(\omega^*)$  is the most complete state space that it is common knowledge that everyone is aware of at  $\omega^*$ .<sup>16</sup> Since we have assumed that  $S^*$  is finite, Theorem 3 in Galanis (2007) states that there is a public event  $E \subseteq \Omega^\wedge(\omega^*)$  such that  $\omega_{\Omega^\wedge(\omega^*)}^* \subseteq E \subseteq E^*$ . The proof of Theorem 4 in Galanis (2007) shows that  $E$  is partitioned by  $P^i$ , for each  $i$ . By adding up we have that for each  $i$ ,  $\sum_{\omega \in E} \pi|_{\Omega(\omega)} b^i(\omega_{S_0}) > 0$ . By adding over all agents we have  $\sum_{\omega \in E} \pi|_{\Omega(\omega)} \sum_{i \in I} b^i(\omega_{S_0}) > 0$ . Since  $\sum_{i \in I} b^i(\omega_{S_0}) = 0$  for all  $\omega \in E$ , we have a contradiction.  $\square$

*Proof of Theorem 3.* Suppose there is a common prior  $\pi$ . Fix agent  $i$  and a state  $\omega^* \in S^*$  such that  $\pi(\omega^*) > 0$ . Then,  $\pi(P^i(\omega^*)) > 0$  and  $\sum_{\omega \in P^i(\omega^*)} b^i(\omega_{S_0}) \pi|_{\Omega^i(\omega^*)}(\omega) > 0$ .

Conditional independence and generalized reflexivity imply that for  $\omega, \omega' \in P^i(\omega^*)$

<sup>16</sup>For  $n \geq 2$ , let

$$\Omega^{i_1 \dots i_n}(\omega^*) = \bigwedge_{\omega' \in P^{i_1 \dots i_n}(\omega^*)} \Omega^{i_n}(\omega')$$

and define  $\Omega^\wedge(\omega^*)$  to be the meet of all state spaces  $\Omega^{i_1 \dots i_n}(\omega^*)$ , for any sequence  $i_1, \dots, i_n$ ,  $n \in \mathbb{N}$ :

$$\Omega^\wedge(\omega^*) = \bigwedge_{\substack{i_1 \dots i_n \\ n \in \mathbb{N}}} \Omega^{i_1 \dots i_n}(\omega^*).$$

Lemma 3 in Galanis (2007) shows that every agent is aware of  $\Omega^\wedge(\omega^*)$  and this fact is common knowledge at  $\omega^*$ . Moreover,  $\Omega^\wedge(\omega^*)$  is the most complete universal event with this property.

such that  $\pi|_{\Omega^i(\omega^*)}(\omega), \pi|_{\Omega^i(\omega^*)}(\omega') > 0$ , we have

$$\frac{\pi(\mathcal{E}^i(\omega^*) \cap \omega^{S^*})}{\pi|_{\Omega^i(\omega^*)}(\omega)} = \frac{\pi(\mathcal{E}^i(\omega^*) \cap \omega'_{S^*})}{\pi|_{\Omega^i(\omega^*)}(\omega')} > 0.$$

Multiplying by that number we have that,

$$\begin{aligned} \sum_{\omega \in P^i(\omega^*)} b^i(\omega_{S_0}) \pi(\mathcal{E}^i(\omega^*) \cap \omega^{S^*}) > 0 &\implies \\ \sum_{\omega \in P^i(\omega^*)} b^i(\omega_{S_0}) \sum_{\omega_1^* \in \mathcal{E}^i(\omega^*) \cap \omega^{S^*}} \pi(\omega_1^*) > 0. \end{aligned}$$

Since  $\{\omega_1^*\}_{S_0} = \omega_{S_0}$  for all  $\omega_1^* \in \omega^{S^*}$  we have

$$\begin{aligned} \sum_{\omega \in P^i(\omega^*)} \sum_{\omega_1^* \in \mathcal{E}^i(\omega^*) \cap \omega^{S^*}} b^i(\{\omega_1^*\}_{S_0}) \pi(\omega_1^*) > 0 &\implies \\ \sum_{\omega_1^* \in (P^i(\omega^*))_{S^*} \cap \mathcal{E}^i(\omega^*)} b^i(\{\omega_1^*\}_{S_0}) \pi(\omega_1^*) > 0. \end{aligned}$$

From the proof of the first part of Theorem 1 we know that  $\{(P^i(\omega^*))_{S^*} \cap \mathcal{E}^i(\omega^*)\}_{\omega^* \in S^*}$  generates a partition of  $S^*$ . By adding all elements of the partition we have that

$$\sum_{\omega^* \in S^*} b^i(\{\omega^*\}_{S_0}) \pi(\omega_1^*) > 0.$$

By adding all agents and since  $\sum_{i \in I} b^i(\omega) = 0$  for each  $\omega \in S_0$ , we derive the contradiction.

For the last claim of the theorem, the counter example is provided in example 4.  $\square$

*Proof of Theorem 4.* The proof is similar to the proof of Theorem 3 in Geanakoplos (1989). Let  $\{f^j\}_{j \in J}$  be an equilibrium and look at  $i$ 's one agent decision problem that is induced when the strategy of each  $j \neq i$  is fixed. Because of PPOP, assumption 1 is satisfied. If agent  $i$  was fully aware but had no information at all, his optimal action would be  $z^i$  and his ex ante payoff would be  $\bar{u}^i$ . Since agent  $i$  is strongly more informed, his awareness is more informative, and satisfies either nested awareness or conditional independence, by Theorem 1, his ex ante payoff is weakly higher than  $\bar{u}^i$ . Since this is true for all agents, by hypothesis,  $f^i(\omega^*) = z^i$ , for all  $\omega^* \in S^*$ , for all  $i \in I$ .  $\square$

*Proof of Theorem 5.* Note that for each  $S$ ,  $\alpha_1(S) \supseteq \beta_1(S) \supseteq \dots \alpha_t(S) \supseteq \beta_t(S) \supseteq \dots$ . Moreover,  $\alpha_t(S^*), \beta_t(S^*) \neq \emptyset$  for all  $t$ . Since the union of all state spaces  $\Sigma$  is finite, there exists  $t$  such that  $\alpha_t(S) = \beta_t(S)$  for all  $S \in \mathcal{S}$ . Consider a state space  $S$  such that  $\alpha_t(S) = \beta_t(S) \neq \emptyset$  and  $S' \prec S$  implies  $\alpha_t(S') = \beta_t(S') = \emptyset$ . Such a  $S$  exists because of the finiteness of  $\Sigma$ . We then have that for all  $\omega \in \alpha_t(S)$ ,  $P_t^\alpha(\omega), P_t^\beta(\omega) \subseteq \alpha_t(S)$ . That is,  $\alpha_t(S) = \beta_t(S)$  is partitioned by  $P_t^\alpha$  and  $P_t^\beta$ . By adding up, we have that

$$\pi(\beta_t(S)) q_t^\beta(\omega^*) = \pi(\beta_t(S) \cap A_S).$$

$$\pi(\alpha_t(S))q_t^\alpha(\omega^*) = \pi(\alpha_t(S) \cap A_S).$$

Hence,  $q_t^\alpha(\omega^*) = q_t^\beta(\omega^*)$ . Note that both agents may be more aware than  $S$ .

For the second claim, if  $\Omega_{t-1}^\alpha(\omega^*) \wedge \Omega_{t-1}^\beta(\omega^*) \prec \Omega_t^\alpha(\omega^*) \wedge \Omega_{t-1}^\beta(\omega^*)$  then  $\Omega_{t-1}^\alpha(\omega^*) \prec \Omega_t^\alpha(\omega^*)$  and  $\alpha$  updates at  $t$ . Suppose that at  $t$  agent  $\alpha$  updates his awareness, so that  $\Omega_{t-1}^\alpha(\omega^*) \prec \Omega_t^\alpha(\omega^*) = S$ . Suppose that it is not the case that  $\Omega_{t-1}^\alpha(\omega^*) \wedge \Omega_{t-1}^\beta(\omega^*) \prec \Omega_t^\alpha(\omega^*) \wedge \Omega_{t-1}^\beta(\omega^*)$ . Because  $\Omega_{t-1}^\alpha(\omega^*) \preceq \Omega_t^\alpha(\omega^*)$  and  $\Omega_{t-1}^\alpha(\omega^*) \wedge \Omega_{t-1}^\beta(\omega^*) \preceq \Omega_t^\alpha(\omega^*) \wedge \Omega_{t-1}^\beta(\omega^*)$ , we must have  $\Omega_{t-1}^\alpha(\omega^*) \wedge \Omega_{t-1}^\beta(\omega^*) \succeq \Omega_t^\alpha(\omega^*) \wedge \Omega_{t-1}^\beta(\omega^*)$ . Hence,  $\Omega_{t-1}^\alpha(\omega^*) \wedge \Omega_{t-1}^\beta(\omega^*) = \Omega_t^\alpha(\omega^*) \wedge \Omega_{t-1}^\beta(\omega^*) = S \wedge \Omega_{t-1}^\beta(\omega^*) \equiv S'$ .

We next show that  $P_{t-1}^\beta(\omega_S^*) = P_{t-1}^\beta(\omega_{S'}^*)$ . Since  $S' \preceq S$  and from Projections Preserve Ignorance,  $\Omega_{t-1}^\beta(\omega_S^*) \succeq \Omega_{t-1}^\beta(\omega_{S'}^*)$ . Also,  $\Omega_{t-1}^\beta(\omega^*) \succeq \Omega_{t-1}^\beta(\omega_S^*)$  implies  $S' = \Omega_{t-1}^\beta(\omega^*) \wedge S \succeq \Omega_{t-1}^\beta(\omega_S^*) \wedge S = \Omega_{t-1}^\beta(\omega_S^*)$ . Again by PPI, we have  $\Omega_{t-1}^\beta(\omega_{S'}^*) \succeq \Omega_{t-1}^\beta(\omega_{\Omega_{t-1}^\beta(\omega_S^*)}^*) = \Omega_{t-1}^\beta(\omega_S^*)$ . The last equality holds from Generalized Reflexivity and Stationarity. Finally, Stationarity and  $\Omega_{t-1}^\beta(\omega_S^*) = \Omega_{t-1}^\beta(\omega_{S'}^*)$  imply  $P_{t-1}^\beta(\omega_S^*) = P_{t-1}^\beta(\omega_{S'}^*)$ .

Because  $\Omega_{t-1}^\alpha(\omega^*)$  rationalizes  $\beta$ 's announcement at  $t-2$ , we have that  $\omega_{\Omega_{t-1}^\alpha(\omega^*)}^* \in \beta_{t-2}(\Omega_{t-1}^\alpha(\omega^*))$ . Moreover,  $\omega_{\Omega_{t-1}^\alpha(\omega^*)}^* \in \alpha_{t-1}(\Omega_{t-1}^\alpha(\omega^*))$  because  $P_{t-1}^\alpha(\omega^*) = P_{t-1}^\alpha(\omega_{\Omega_{t-1}^\alpha(\omega^*)}^*)$ . Because  $S \succeq \Omega_{t-1}^\alpha(\omega^*) \succeq S'$  we have  $\Omega_{t-1}^\beta(\omega_S^*) \succeq \Omega_{t-1}^\beta(\omega_{\Omega_{t-1}^\alpha(\omega^*)}^*) \succeq \Omega_{t-1}^\beta(\omega_{S'}^*)$ . Hence,  $\Omega_{t-1}^\beta(\omega_S^*) = \Omega_{t-1}^\beta(\omega_{\Omega_{t-1}^\alpha(\omega^*)}^*) = \Omega_{t-1}^\beta(\omega_{S'}^*)$  and  $P_{t-1}^\beta(\omega_S^*) = P_{t-1}^\beta(\omega_{\Omega_{t-1}^\alpha(\omega^*)}^*) = P_{t-1}^\beta(\omega_{S'}^*)$ . This implies that  $\omega_{\Omega_{t-1}^\alpha(\omega^*)}^* \in \beta_{t-1}(\Omega_{t-1}^\alpha(\omega^*))$ . But then,  $P_{t-1}^\alpha(\omega^*) \cap \beta_{t-1}(\Omega_{t-1}^\alpha(\omega^*)) \neq \emptyset$ , contradicting that  $\alpha$  updates at  $t$ .

For the third claim, suppose that at  $t$  agent  $\alpha$  updates his awareness, so that  $\Omega_{t-1}^\alpha(\omega^*) \prec \Omega_t^\alpha(\omega^*) = S$ . Define  $S' = (S \wedge \Omega_{t-1}^\beta(\omega^*)) \vee \Omega_{t-1}^\alpha(\omega^*)$ . From Lemma 6.1 in [Davey and Priestley \(1990\)](#) we have  $S' = (S \wedge \Omega_{t-1}^\beta(\omega^*)) \vee \Omega_{t-1}^\alpha(\omega^*) \preceq (S \vee \Omega_{t-1}^\alpha(\omega^*)) \wedge (\Omega_{t-1}^\alpha(\omega^*) \vee \Omega_{t-1}^\beta(\omega^*)) = S \wedge (\Omega_{t-1}^\beta(\omega^*) \vee \Omega_{t-1}^\alpha(\omega^*)) \preceq \Omega_{t-1}^\alpha(\omega^*) \vee \Omega_{t-1}^\beta(\omega^*)$ . Hence,  $S' \vee \Omega_{t-1}^\beta(\omega^*) \preceq \Omega_{t-1}^\alpha(\omega^*) \vee \Omega_{t-1}^\beta(\omega^*)$ . Moreover,  $S' \vee \Omega_{t-1}^\beta(\omega^*) \succeq \Omega_{t-1}^\alpha(\omega^*) \vee \Omega_{t-1}^\beta(\omega^*)$ .

We next show that  $\Omega_{t-1}^\beta(\omega_S^*) = \Omega_{t-1}^\beta(\omega_{S'}^*)$ . Since  $S' \preceq S$  and from Projections Preserve Ignorance,  $\Omega_{t-1}^\beta(\omega_S^*) \succeq \Omega_{t-1}^\beta(\omega_{S'}^*)$ . Also,  $\Omega_{t-1}^\beta(\omega^*) \succeq \Omega_{t-1}^\beta(\omega_S^*)$  implies  $\Omega_{t-1}^\beta(\omega^*) \wedge S \succeq \Omega_{t-1}^\beta(\omega_S^*) \wedge S = \Omega_{t-1}^\beta(\omega_S^*)$  and  $S' = (\Omega_{t-1}^\beta(\omega^*) \wedge S) \vee \Omega_{t-1}^\alpha(\omega^*) \succeq \Omega_{t-1}^\beta(\omega_S^*) \vee \Omega_{t-1}^\alpha(\omega^*) \succeq \Omega_{t-1}^\beta(\omega_S^*)$ . Again by PPI, we have  $\Omega_{t-1}^\beta(\omega_{S'}^*) \succeq \Omega_{t-1}^\beta(\omega_{\Omega_{t-1}^\beta(\omega_S^*)}^*) = \Omega_{t-1}^\beta(\omega_S^*)$ . The last equality holds from Generalized Reflexivity and Stationarity. Finally, Stationarity and  $\Omega_{t-1}^\beta(\omega_S^*) = \Omega_{t-1}^\beta(\omega_{S'}^*)$  imply  $P_{t-1}^\beta(\omega_S^*) = P_{t-1}^\beta(\omega_{S'}^*)$ .

Because  $S$  rationalizes  $\beta$ 's announcement at  $t-1$  and from Generalized Reflexivity we have that

$$q_{t-1}^\beta(\omega^*) = \frac{\pi(P_{t-1}^\beta(\omega_S^*) \cap A_{\Omega_{t-1}^\beta(\omega_S^*)})}{\pi(P_{t-1}^\beta(\omega_S^*))}.$$

From the proof of the second claim we have that  $\omega_{\Omega_{t-1}^\alpha(\omega^*)}^* \in \beta_{t-1}(\Omega_{t-1}^\alpha(\omega^*))$ , which implies (because  $\Omega_{t-1}^\alpha(\omega^*) \preceq S'$ ) that  $\omega_{S'}^* \in \beta_{t-1}(S')$ . Because  $P_{t-1}^\beta(\omega_S^*) = P_{t-1}^\beta(\omega_{S'}^*)$  and  $S \succeq S'$  is minimal, we have  $S = S'$ . Hence,  $\Omega_t^\alpha(\omega^*) \vee \Omega_{t-1}^\beta(\omega^*) = \Omega_{t-1}^\alpha(\omega^*) \vee$

$\Omega_{t-1}^\beta(\omega^*)$ . With similar arguments we can show for agent  $\beta$  that  $\Omega_t^\alpha(\omega^*) \vee \Omega_t^\beta(\omega^*) = \Omega_t^\alpha(\omega^*) \vee \Omega_{t-1}^\beta(\omega^*)$ . Hence,  $\Omega_{t-1}^\alpha(\omega^*) \vee \Omega_{t-1}^\beta(\omega^*) = \Omega_t^\alpha(\omega^*) \vee \Omega_t^\beta(\omega^*)$ .  $\square$

## B Counter examples

**Example 1.** *Both agents satisfy conditional independence, agent 2 is weakly (but not strongly) more informed, his awareness is nested and more informative and he is strictly worse off. There are two basic questions,  $q$  and  $r$ . Agent 2 is only aware of the first question, while agent 1 is always fully aware. Since both agents have constant awareness, they satisfy conditional independence and nested awareness. Agent 1 always learns the answer to the question  $r$ , while both never learn the answer to question  $q$ . There are two state spaces,  $S^* = \{\omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*\}$ ,*

$$\begin{aligned}\omega_1^* &= (q_y, r_n), & \pi(\omega_1^*) &= 3/8, \\ \omega_2^* &= (q_n, r_y), & \pi(\omega_2^*) &= 3/8, \\ \omega_3^* &= (q_y, r_y), & \pi(\omega_3^*) &= 1/8, \\ \omega_4^* &= (q_n, r_n), & \pi(\omega_4^*) &= 1/8,\end{aligned}$$

and  $S_0 = \{\omega_1, \omega_2\}$ , where  $\omega_1 = (q_y)$  and  $\omega_2 = (q_n)$ . There are two actions,  $B$  and  $NB$ . We have  $u(\omega_1, NB) = 1$ ,  $u(\omega_1, B) = 1$ ,  $u(\omega_2, NB) = 1$ ,  $u(\omega_2, B) = -1$ . Agent 2 has the trivial partition, so for all  $\omega \in S^* \cup S_0$ ,  $P^2(\omega) = \{\omega_1, \omega_2\}$ . Agent 1's possibility correspondence is as follows:

$$\begin{aligned}P^1(\omega_1^*) &= P^2(\omega_4^*) = \{\omega_1^*, \omega_4^*\}, \\ P^1(\omega_2^*) &= P^2(\omega_3^*) = \{\omega_2^*, \omega_3^*\}, \\ P^1(\omega_1) &= P^2(\omega_2) = \{\omega_1, \omega_2\}.\end{aligned}$$

Agent 2 is indifferent between both actions and his expected utility is 0. Agent 1 chooses action  $B$  at  $\omega_1^*$ ,  $\omega_4^*$  and action  $NB$  at  $\omega_2^*$ ,  $\omega_3^*$ . His expected utility is  $1/2$ .

**Example 2.** *In this example agent 2 is strictly worse off although he is strongly more informed, his awareness is more informed but not nested and he fails conditional independence, whereas agent 1 satisfies conditional independence.*

*The setting is similar to that of the main example. There are three basic questions  $p$ ,  $q$  and  $r$ . Agent 1 is always fully aware but has no information. Hence, he satisfies conditional independence. The full state space  $S^*$  contains three states:*

$$\begin{aligned}\omega_1^* &= (p_l, q_y, r_n), & \pi(\omega_1^*) &= 0.3, \\ \omega_2^* &= (p_m, q_n, r_y), & \pi(\omega_2^*) &= 0.3, \\ \omega_3^* &= (p_h, q_y, r_y), & \pi(\omega_3^*) &= 0.4,\end{aligned}$$

There are three other state spaces:  $S_1 = \{\omega_1, \omega_2, \omega_3\}$  which lacks the  $r$  dimension,  $S_2 = \{\omega_4, \omega_5, \omega_6\}$  which lacks the  $q$  dimension, and  $S_3 = \{\omega_7, \omega_8, \omega_9\}$  which lacks both  $q$  and  $r$ . For example,  $\omega_1 = (p_l, q_y)$ ,  $\omega_5 = (p_m, r_y)$  and  $\omega_9 = (p_h)$ .

Agent 2 is always aware of  $p$ . He is aware of  $q$  at  $\omega_1^*$ ,  $\omega_3^*$  and he is aware of  $r$  at  $\omega_2^*$ ,  $\omega_3^*$ . His information is as follows:  $P^2(\omega_1^*) = \{\omega_1, \omega_3\}$ ,  $P^2(\omega_2^*) = \{\omega_5, \omega_6\}$ ,  $P^2(\omega_3^*) = \{\omega_1^*\}$ .

The payoffs and actions are the same as in the main example. Action  $B$  yields 1 if  $p_h$  and 0 otherwise, while action  $NB$  yields 0 always. Agent 1 has no information so his optimal action is  $NB$ , and his ex ante expected utility is 0. Agent 2 chooses  $B$  always and his ex ante expected utility is -0.2.

**Example 3.** Agent 2 is strongly more informed, his awareness is nested but not more informed and he violates conditional independence. Agent 1 satisfies conditional independence and he is strictly better off.

There are two state spaces,  $S_0 = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ ,  $S^* = \{\omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*\}$ , where  $S_0 \preceq S^*$ . Each  $\omega_i^*$  projects to  $\omega_i$ . Agent 1's information is such that  $P^1(\omega_i^*) = P^1(\omega_i) = \{\omega_1^*, \omega_2^*, \omega_3^*\}$  for  $i = 1, 2, 3$ ,  $P^1(\omega_4^*) = P^1(\omega_4) = \{\omega_4\}$ . For agent 2 we have  $P^2(\omega_1^*) = P^2(\omega_1) = P^2(\omega_2) = \{\omega_1, \omega_2\}$ ,  $P^2(\omega_2^*) = \{\omega_2^*\}$ ,  $P^2(\omega_3^*) = P^2(\omega_3) = \{\omega_3\}$ ,  $P^2(\omega_4^*) = P^2(\omega_4) = \{\omega_4\}$ . There are two actions,  $B$  and  $NB$ . Action  $B$  yields 1 at  $\omega_2$ , -1 at  $\omega_1$ ,  $\omega_3$ , and 0 at  $\omega_4$ . Action  $NB$  yields 0 always. The prior  $\pi$  is defined as:  $\pi(\omega_1^*) = 0.3$ ,  $\pi(\omega_2^*) = 0.35$ ,  $\pi(\omega_3^*) = 0.3$ ,  $\pi(\omega_4^*) = 0.05$ . Agent 1 chooses  $NB$  always and his ex ante expected utility is 0.25. Agent 2 chooses  $B$  at  $\omega_1^*$ ,  $\omega_2^*$  and  $\omega_4^*$  and  $NB$  at  $\omega_3^*$ . His ex ante expected utility is 0.05.

**Example 4.** We present an example with two agents whose priors satisfy conditional independence, they have no common prior, non-degeneracy is satisfied and there is no trade that ensures positive expected gains at each full state, for both. There are two state spaces,  $S^* = \{\omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*\}$  and  $S_0 = \{\omega_1, \omega_2\}$ , such that  $S_0 \preceq S^*$ ,  $\omega_{1S_0}^* = \omega_{2S_0}^* = \omega_1$  and  $\omega_{3S_0}^* = \omega_{4S_0}^* = \omega_2$ . Agent 1 is always fully aware and  $P^1(\omega^*) = S^*$  for all  $\omega^* \in S^*$ . His prior  $\pi^1$  on  $S^*$  is  $(1/8, 1/2, 2/8, 1/8)$ . In fact, this is the only prior that can generate his posteriors. Agent 2's possibility correspondence is as follows:  $P^2(\omega_1^*) = P^2(\omega_4^*) = S_0$ ,  $P^2(\omega_2^*) = \{\omega_2^*\}$ ,  $P^2(\omega_3^*) = \{\omega_3^*\}$ . His prior assigns 1/4 to each  $\omega^* \in S^*$ . Since  $\pi^1$  cannot generate 2's posteriors, the agents have no common priors. Moreover, the agents' priors satisfy conditional independence. Suppose there is a trade  $b^i : S_0 \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , such that  $\sum_{\omega \in \Omega^i(\omega^*)} t^i(\omega^*, \omega) b^i(\omega_{S_0}) > 0$ , for each  $\omega^* \in S^*$ , for each

*i*. For agent 2 this means that  $b^2(\omega_1), b^2(\omega_2) > 0$ . But since  $\sum_{i \in I} b^i(\omega) = 0$  for each  $\omega \in S_0$ , we have  $b^1(\omega_1), b^1(\omega_2) < 0$ , which implies  $\sum_{\omega \in \Omega^1(\omega_1^*)} t^1(\omega_1^*, \omega) b^1(\omega_{S_0}) < 0$ .

**Example 5.** This is an example of common knowledge trade and common priors, within the standard model and with non-partitional information structures. There are three states  $\{\omega_1, \omega_2, \omega_3\}$  and the prior is such that  $\pi(\omega_1), \pi(\omega_3) = 1/4$ .  $\pi(\omega_2) = 1/2$ . There are two agents. Agent 1 has the trivial partition  $P^1(\omega) = \Omega$  for all  $\omega \in \Omega$ . For agent 2 we have  $P^2(\omega_1) = P^2(\omega_2) = \{\omega_1, \omega_2\}$ ,  $P^2(\omega_3) = \{\omega_2, \omega_3\}$ . Consider the trade  $b^1(\omega_1) = b^1(\omega_3) = 1/4$ ,  $b^1(\omega_2) = -3/16$ ,  $b^2 = -b^1$ . At each state  $\omega \in \Omega$ , both agents expect positive gains. Hence, this is always common knowledge.

**Example 6.** This is an example of an information structure  $P^2$  that is not strongly more informed than  $P^1$ ,  $S_0$  is non degenerate for both  $P^1$  and  $P^2$ , but for any  $u, \pi, C$ , decision problem  $(C, \Sigma, P^2, u, \pi)$  is more valuable than  $(C, \Sigma, P^1, u, \pi)$ . There are four state spaces,  $S_0 = \{\omega\}$ ,  $S_1 = \{\omega_1, \omega_2\}$ ,  $S_2 = \{\omega_3, \omega_4\}$  and  $S^* = \{\omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*\}$ . All states project to  $\omega$ . Moreover,  $P^1(\omega_1^*) = P^1(\omega_2^*) = \{\omega_1\}$ ,  $P^1(\omega_3^*) = P^1(\omega_4^*) = \{\omega_2\}$ ,  $P^2(\omega_1^*) = P^1(\omega_3^*) = \{\omega_3\}$ ,  $P^2(\omega_2^*) = P^1(\omega_4^*) = \{\omega_4\}$ . Neither  $P^1$  or  $P^2$  is strongly more informed. Moreover,  $S_0$  is degenerate for both  $P^1$  and  $P^2$  because each agent always considers only one state to be possible. Yet, for any  $u, \pi, C$ , decision problem  $(C, \Sigma, P^2, u, \pi)$  is more valuable as  $(C, \Sigma, P^1, u, \pi)$ .

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